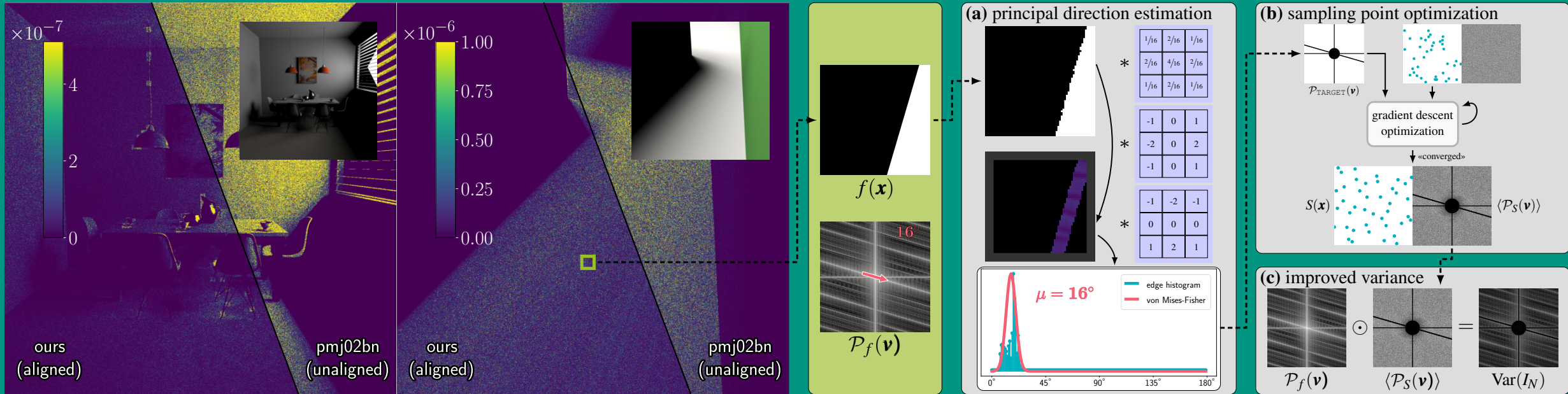




Optimized and Aligned Anisotropic Monte Carlo Sampling Patterns

Mirco Werner, Johannes Hanika, and Carsten Dachsbacher

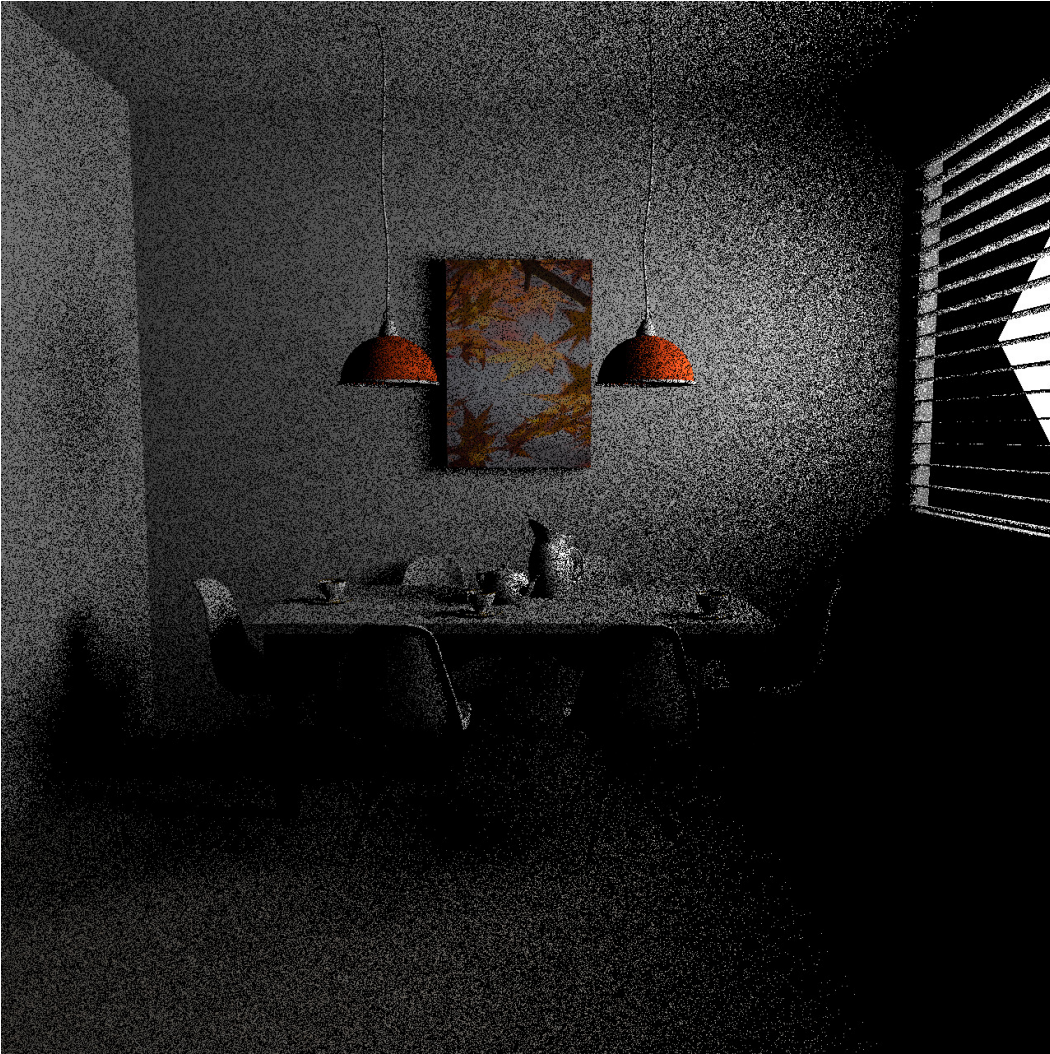
Karlsruhe Institute of Technology | EGSR 2026

Offline Monte Carlo Path Tracing: Direct Illumination



1 spp

Offline Monte Carlo Path Tracing: Direct Illumination

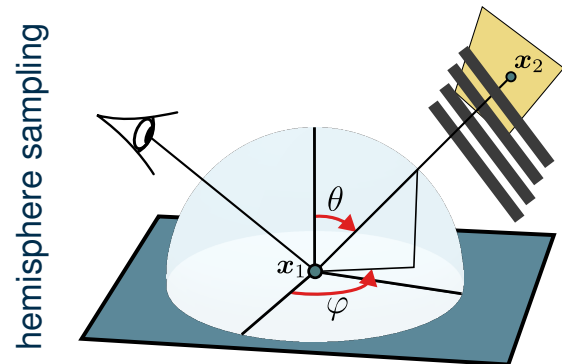
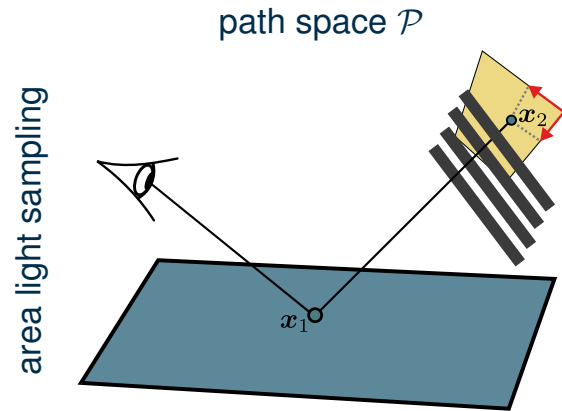


1 spp



reference

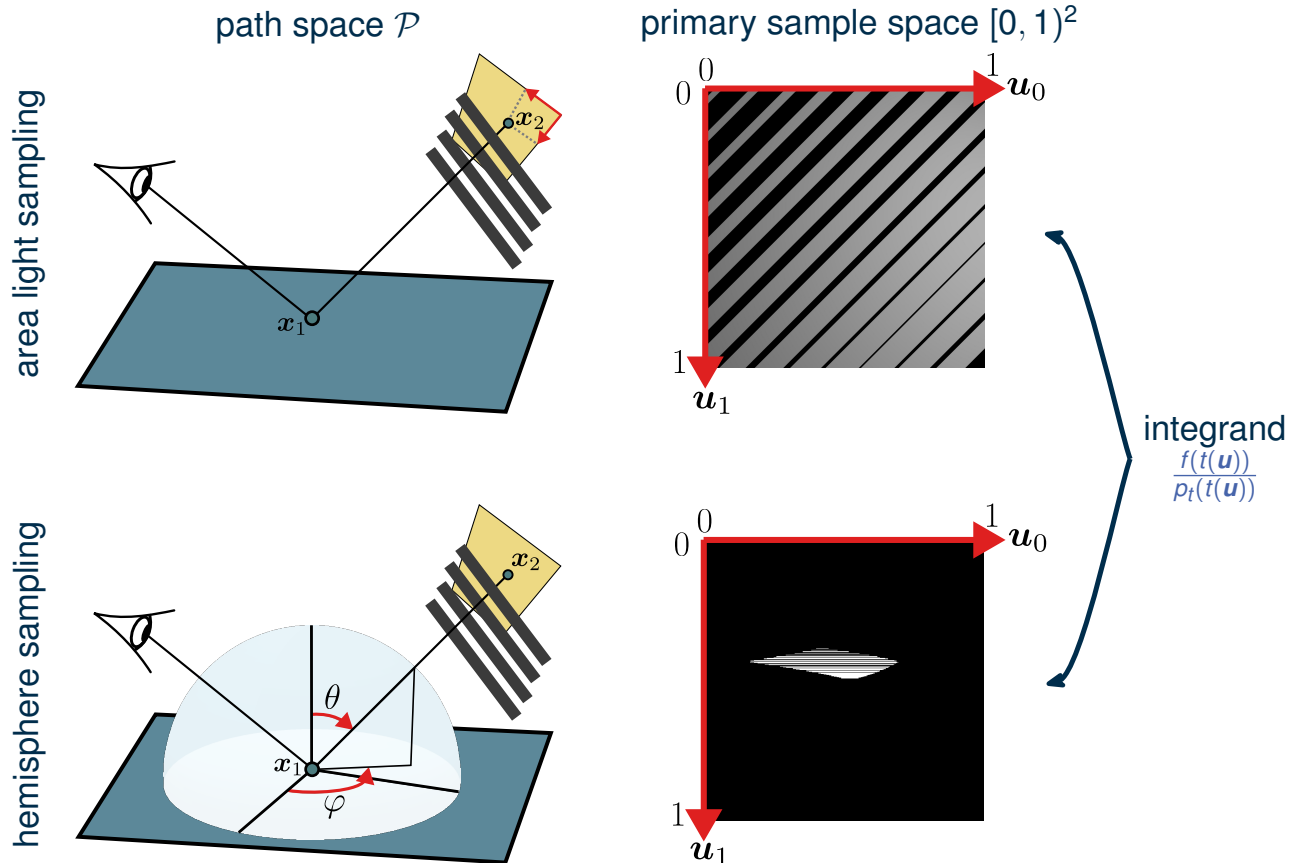
Paths From Random Numbers: Direct Illumination



■ paths $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{P}$

$$I = \int_{\mathcal{P}} f(\mathbf{x}) d\mathbf{x}$$

Paths From Random Numbers: Direct Illumination

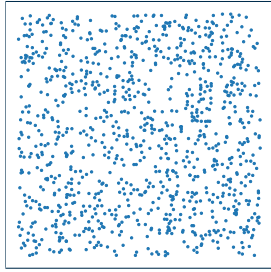


- paths $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] \in \mathcal{P}$
 - from 2D random numbers $\mathbf{u} \in [0, 1]^2$
 - transformation $t : [0, 1]^2 \rightarrow \mathcal{P}$ with $t(\mathbf{u}) = \mathbf{x}$

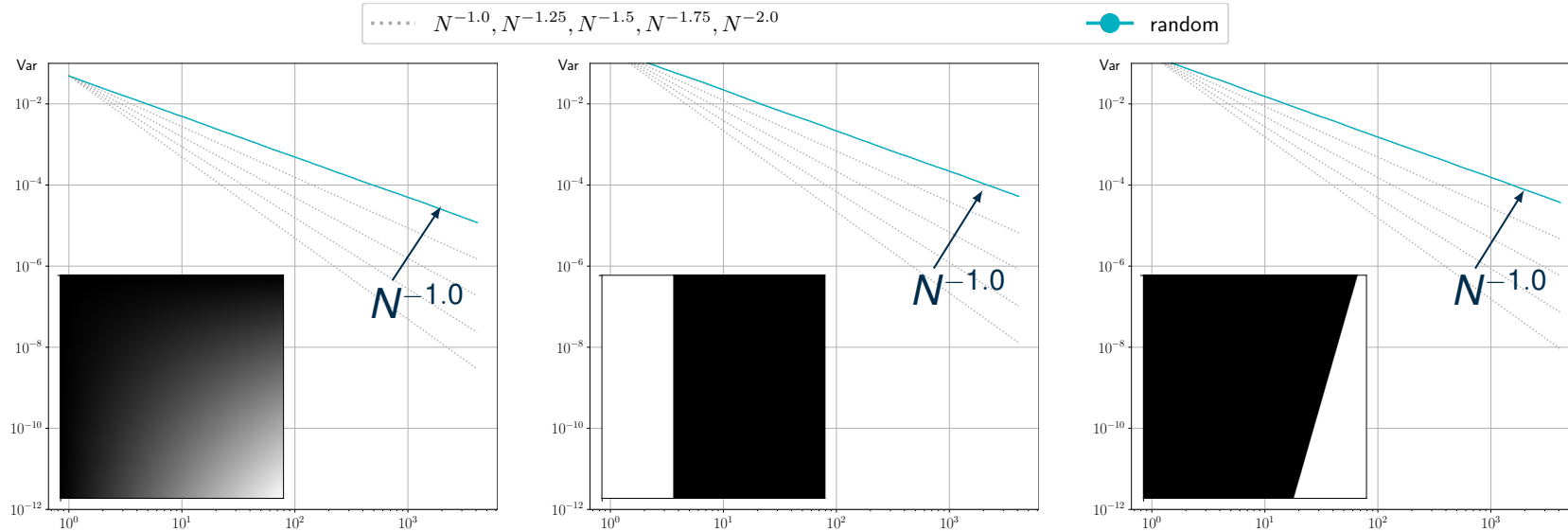
$$\begin{aligned}
 I &= \int_{\mathcal{P}} f(\mathbf{x}) d\mathbf{x} \\
 &= \int_{[0, 1]^2} \frac{f(t(\mathbf{u}))}{p_t(t(\mathbf{u}))} d\mathbf{u}
 \end{aligned}$$

Variance of (quasi-)Monte Carlo Integration

random (Monte Carlo)

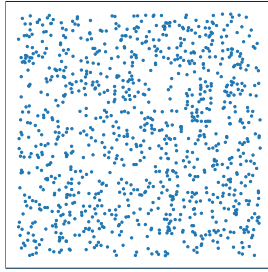


random



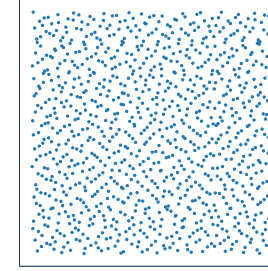
Variance of (quasi-)Monte Carlo Integration

random (Monte Carlo)

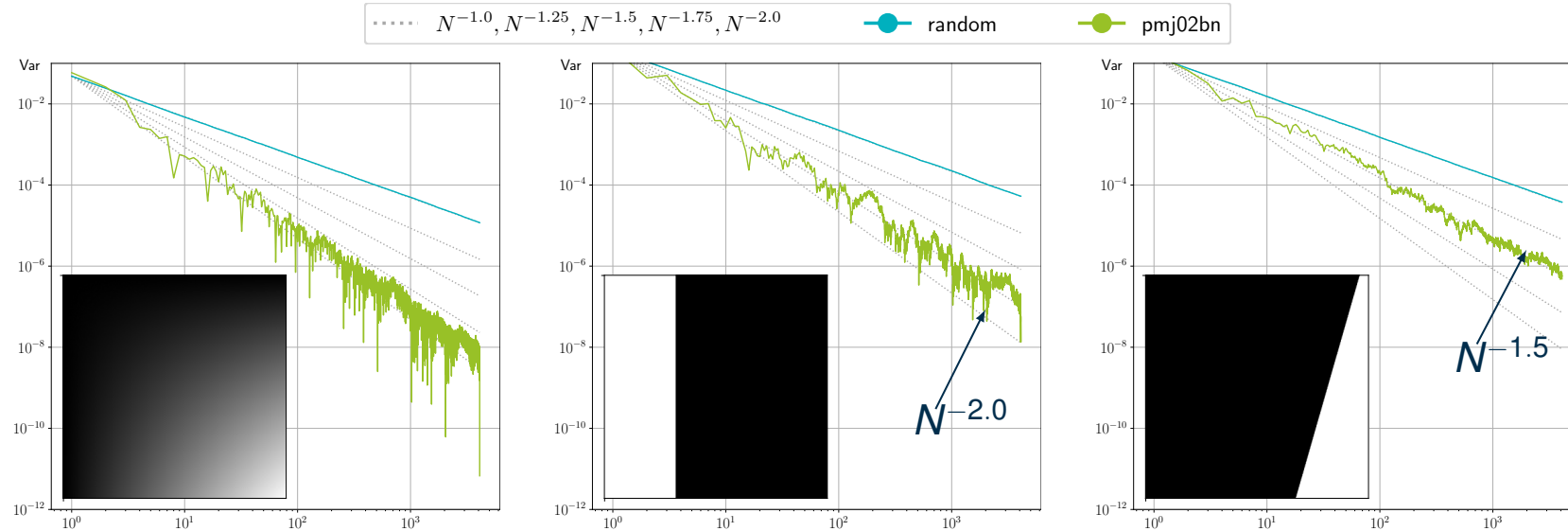


random

correlated (quasi-Monte Carlo)



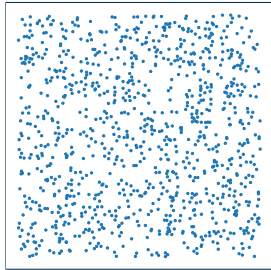
pmj02bn¹



¹progressive multi-jittered (0,2) + blue noise (Christensen et al., 2018)

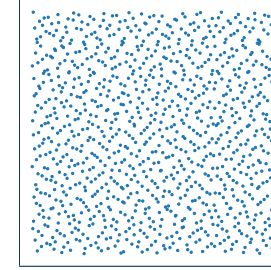
Variance of (quasi-)Monte Carlo Integration

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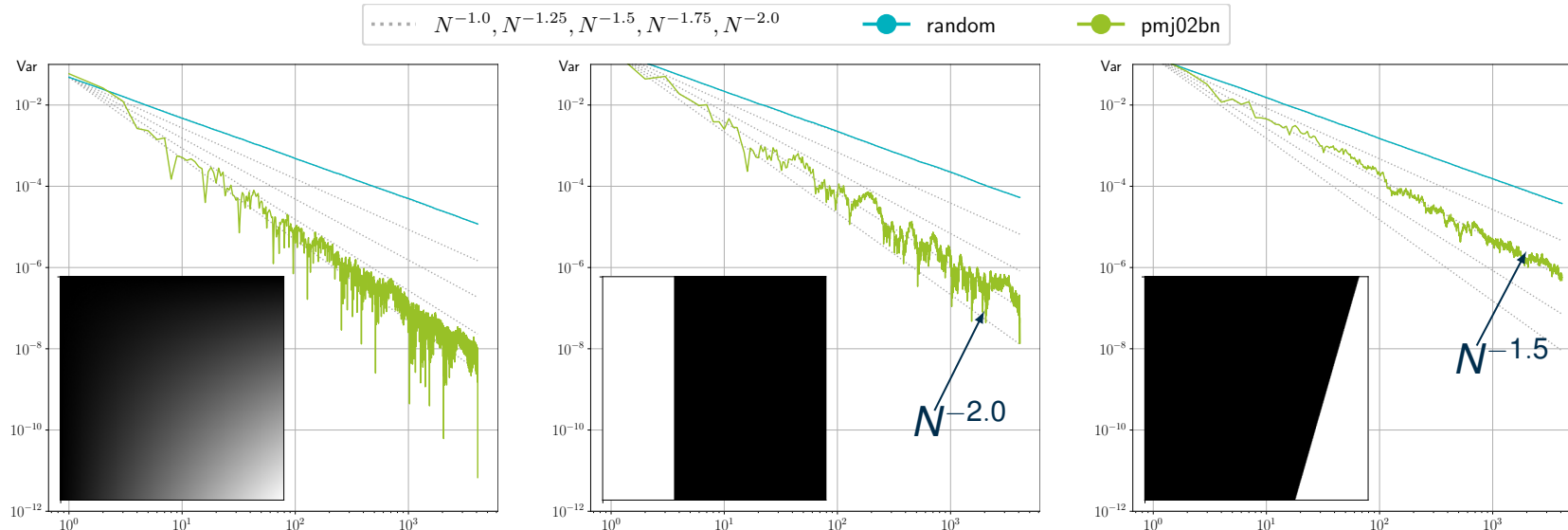


random

correlated (quasi-Monte Carlo)



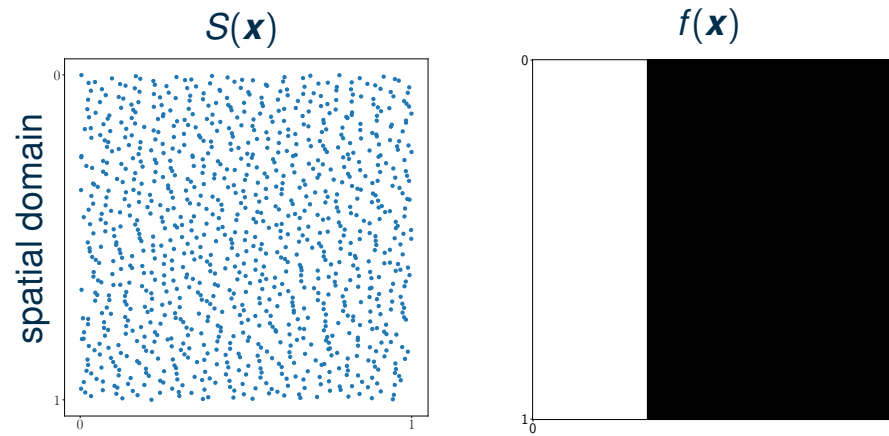
pmj02bn¹



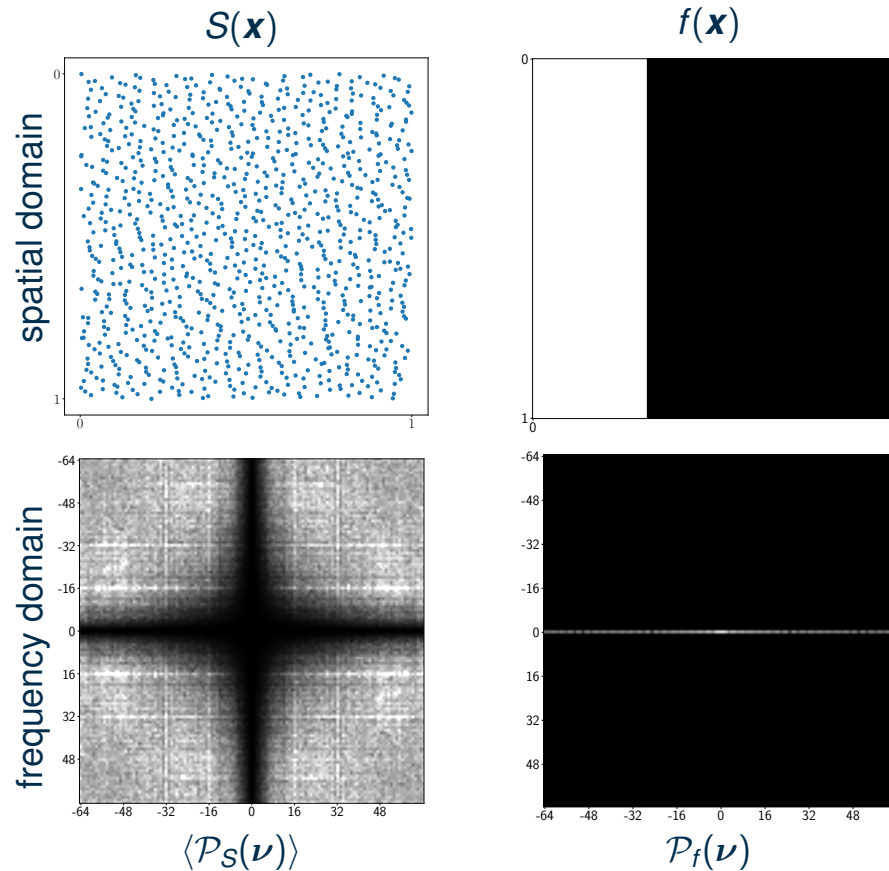
⇒ Can we get better than $\mathcal{O}(N^{-1.5})$ on arbitrarily oriented discontinuous integrands?

¹progressive multi-jittered (0,2) + blue noise (Christensen et al., 2018)

Analyzing Variance of (quasi-)Monte Carlo Integration

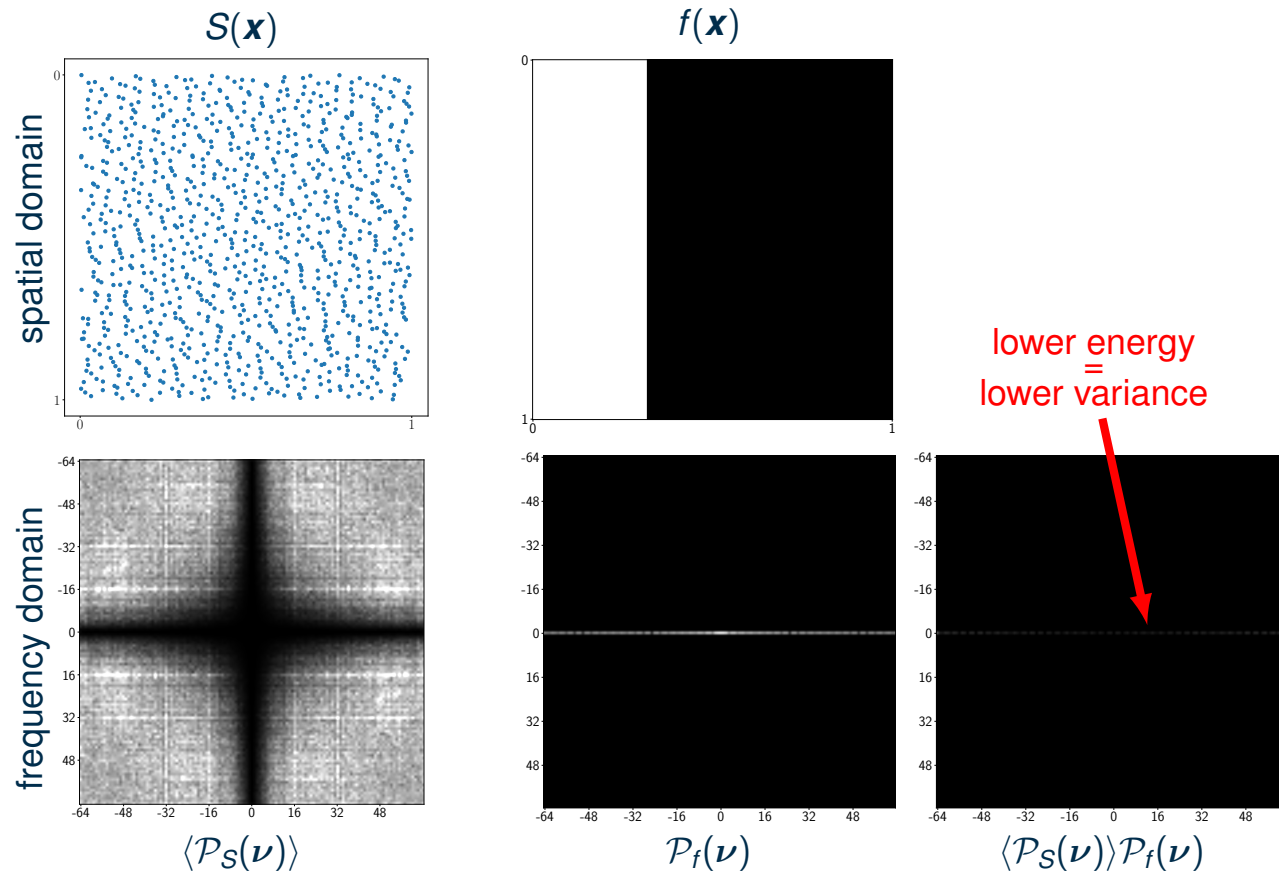


Analyzing Variance of (quasi-)Monte Carlo Integration



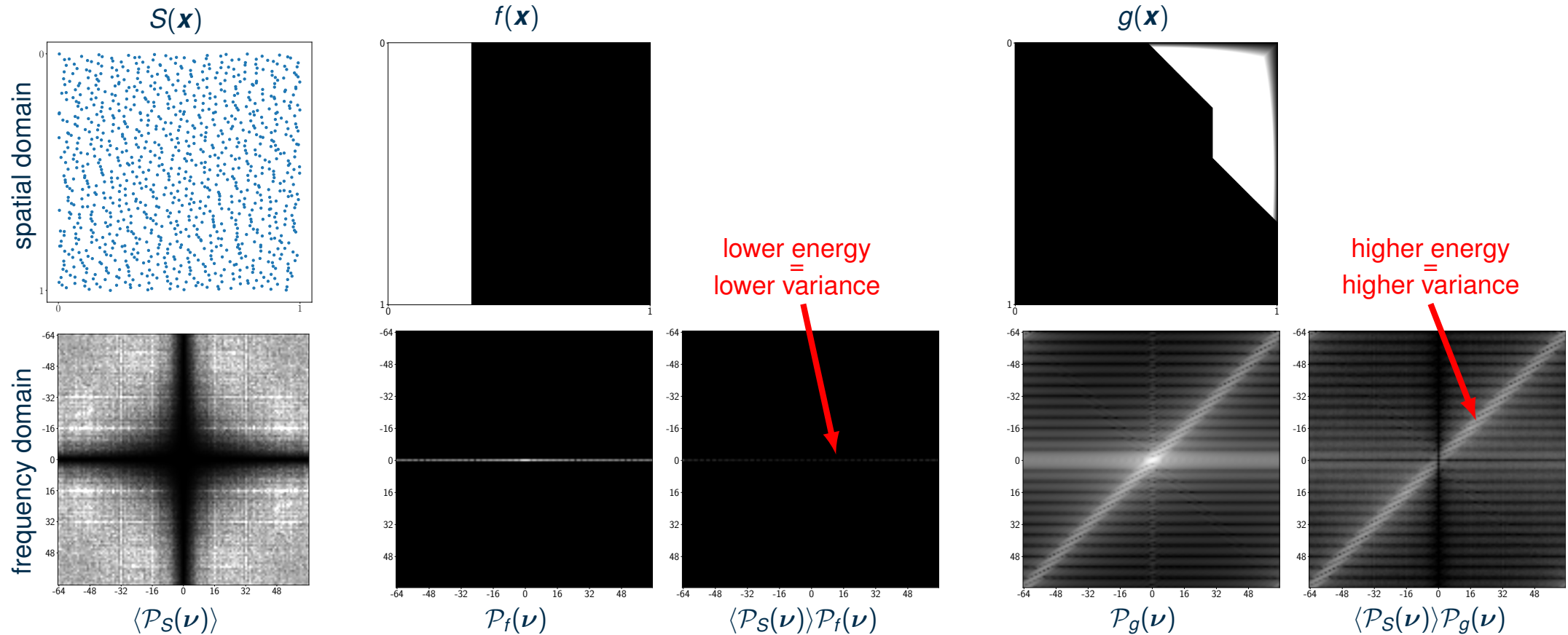
Fourier analysis of Monte Carlo integration (Pilleboue et al., 2015; Subr et al., 2013; Durand, 2011; Singh, Subr, et al., 2020)

Analyzing Variance of (quasi-)Monte Carlo Integration



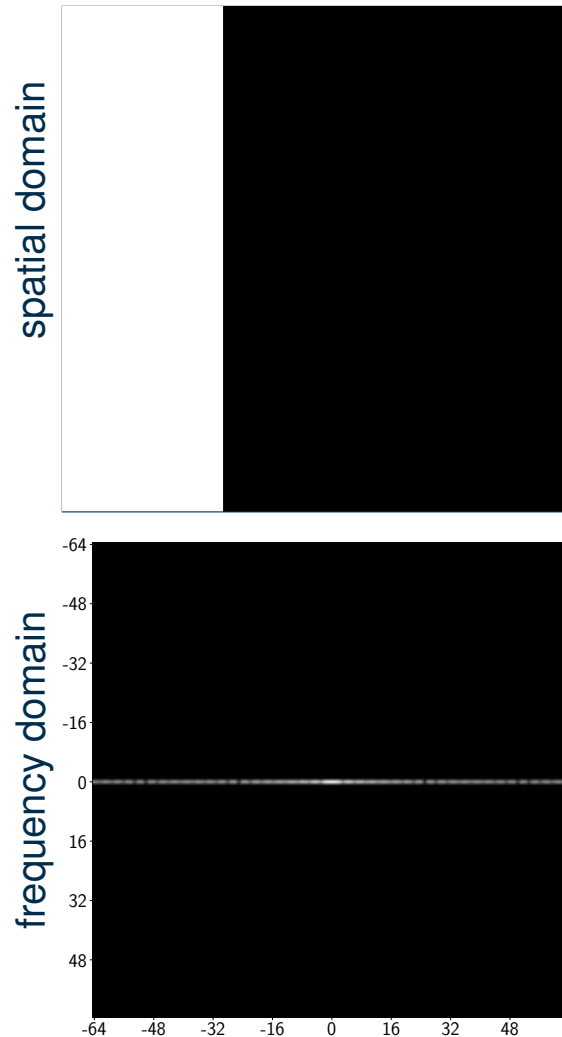
$$\text{Var}(I_N) = \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \langle P_S(\nu) \rangle P_f(\nu)$$

Analyzing Variance of (quasi-)Monte Carlo Integration

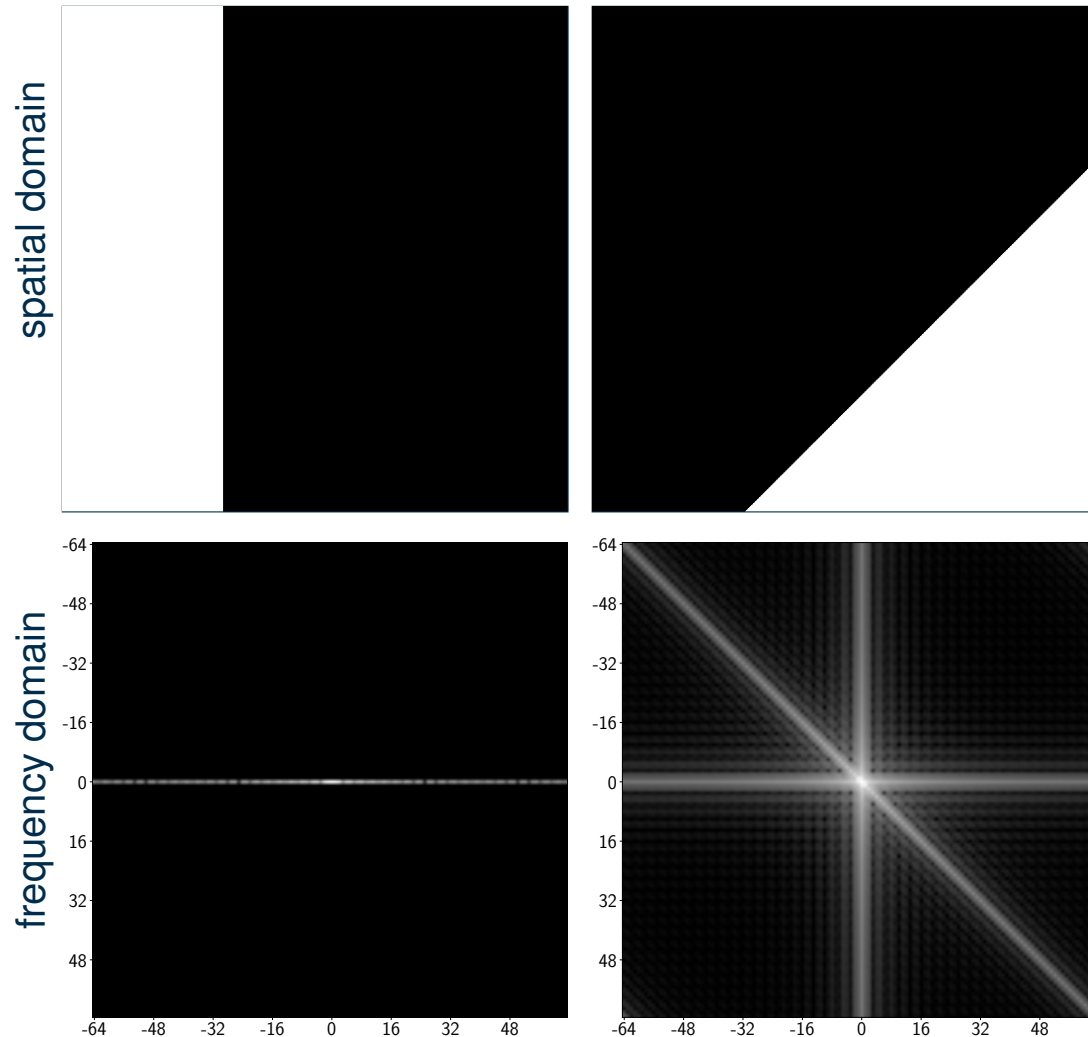


$$\text{Var}(I_N) = \sum_{\nu \in \mathbb{Z}^d \setminus \{0\}} \langle \mathcal{P}_S(\nu) \rangle \mathcal{P}_f(\nu)$$

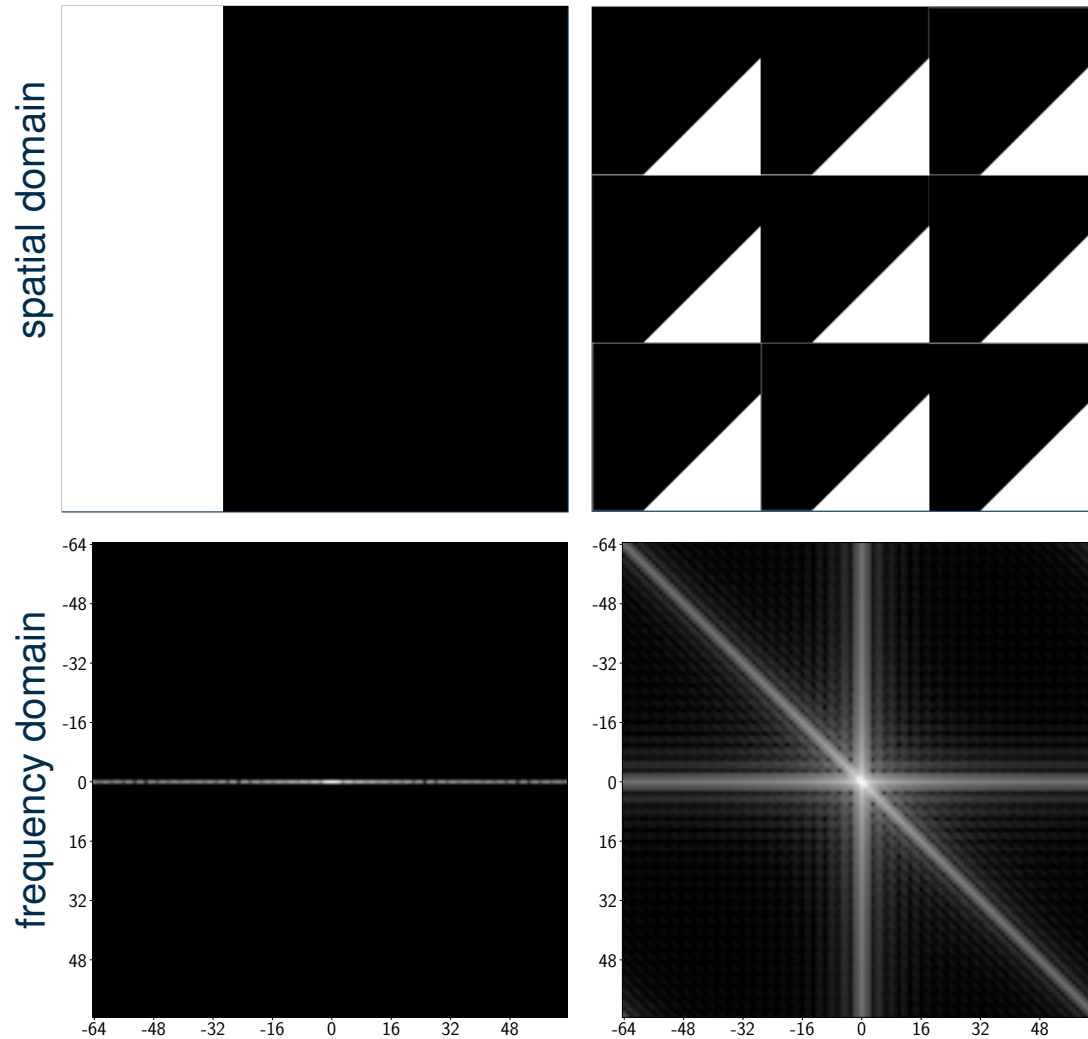
Discontinuous Integrands and Sampling Patterns



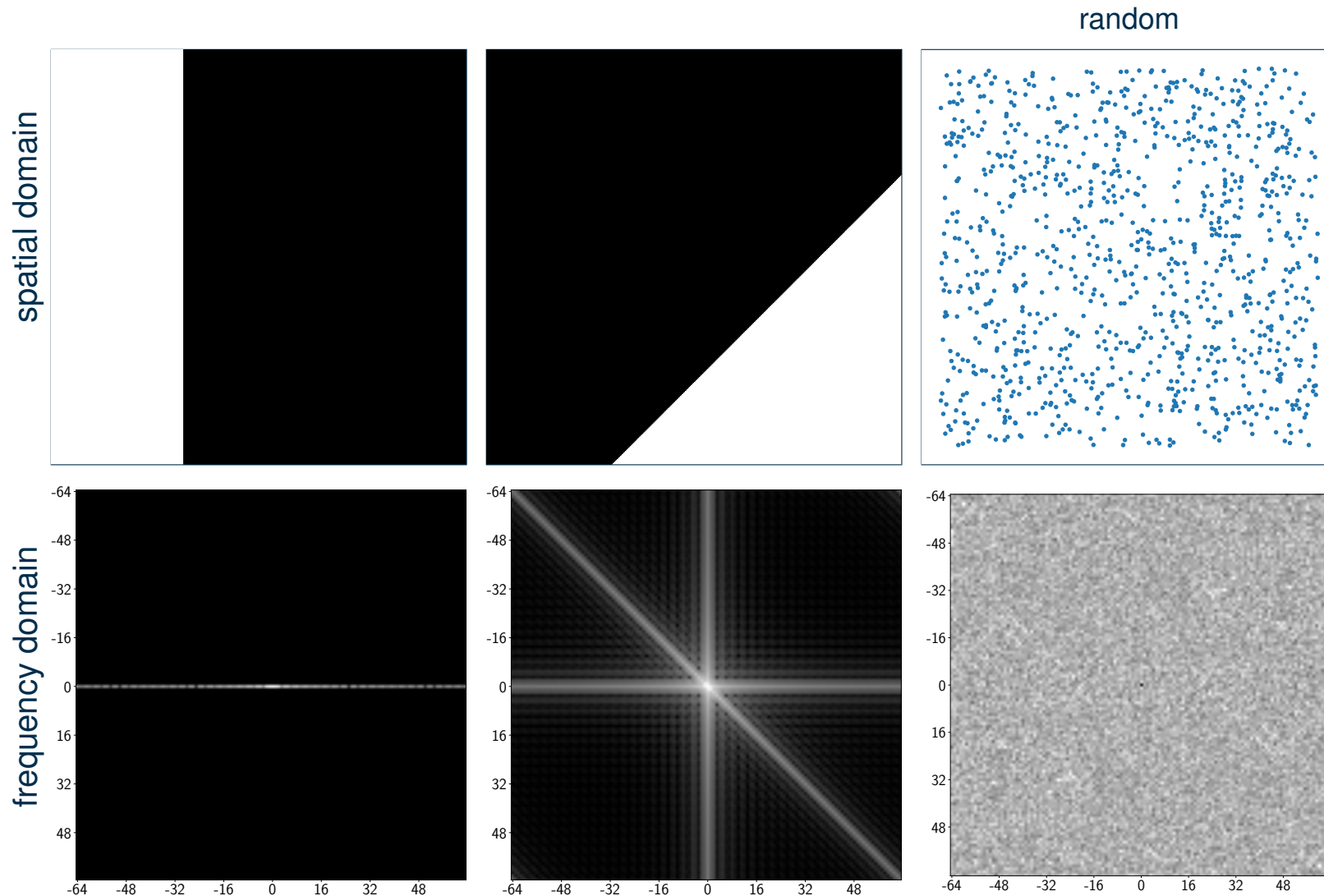
Discontinuous Integrands and Sampling Patterns



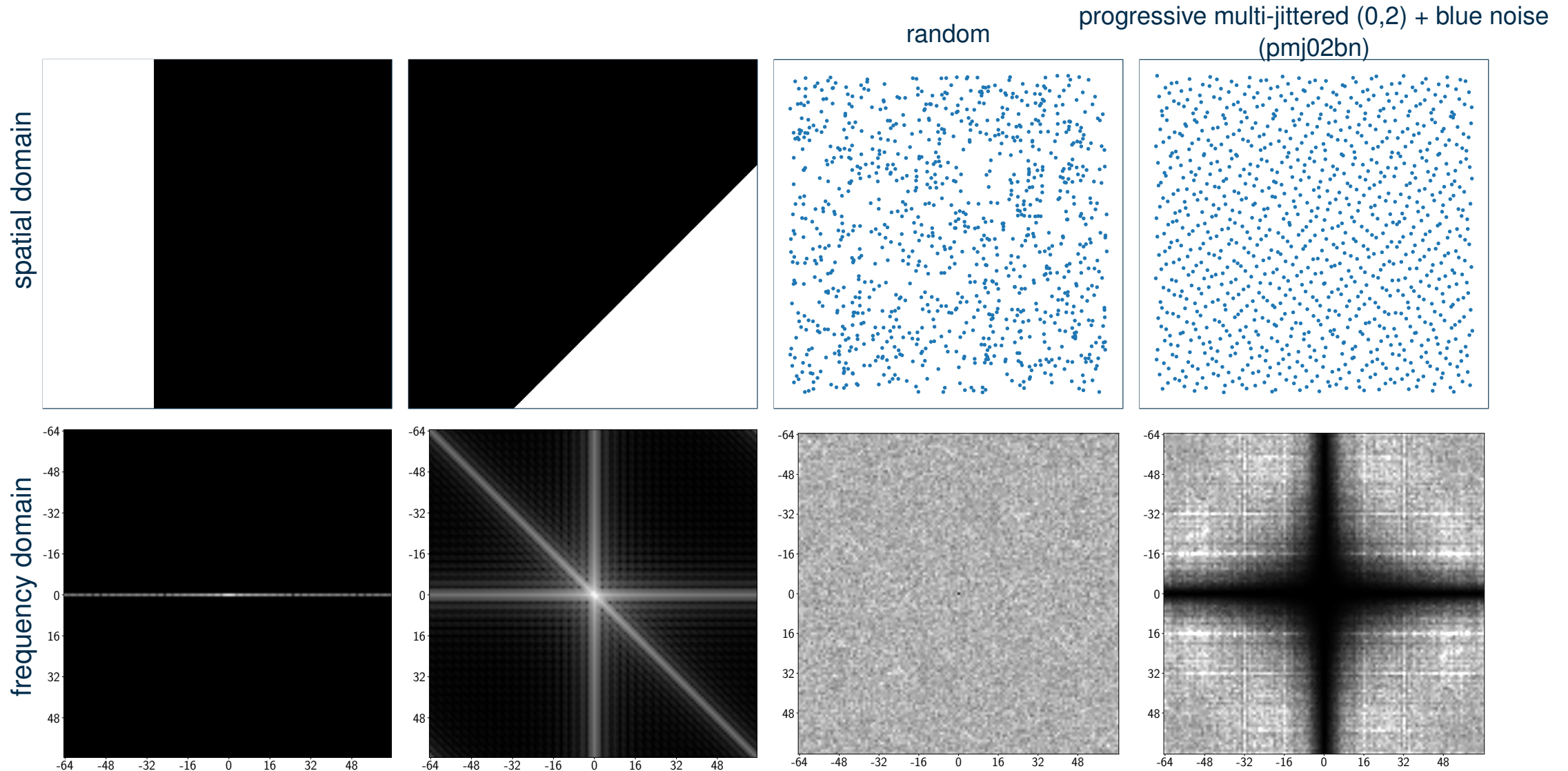
Discontinuous Integrands and Sampling Patterns



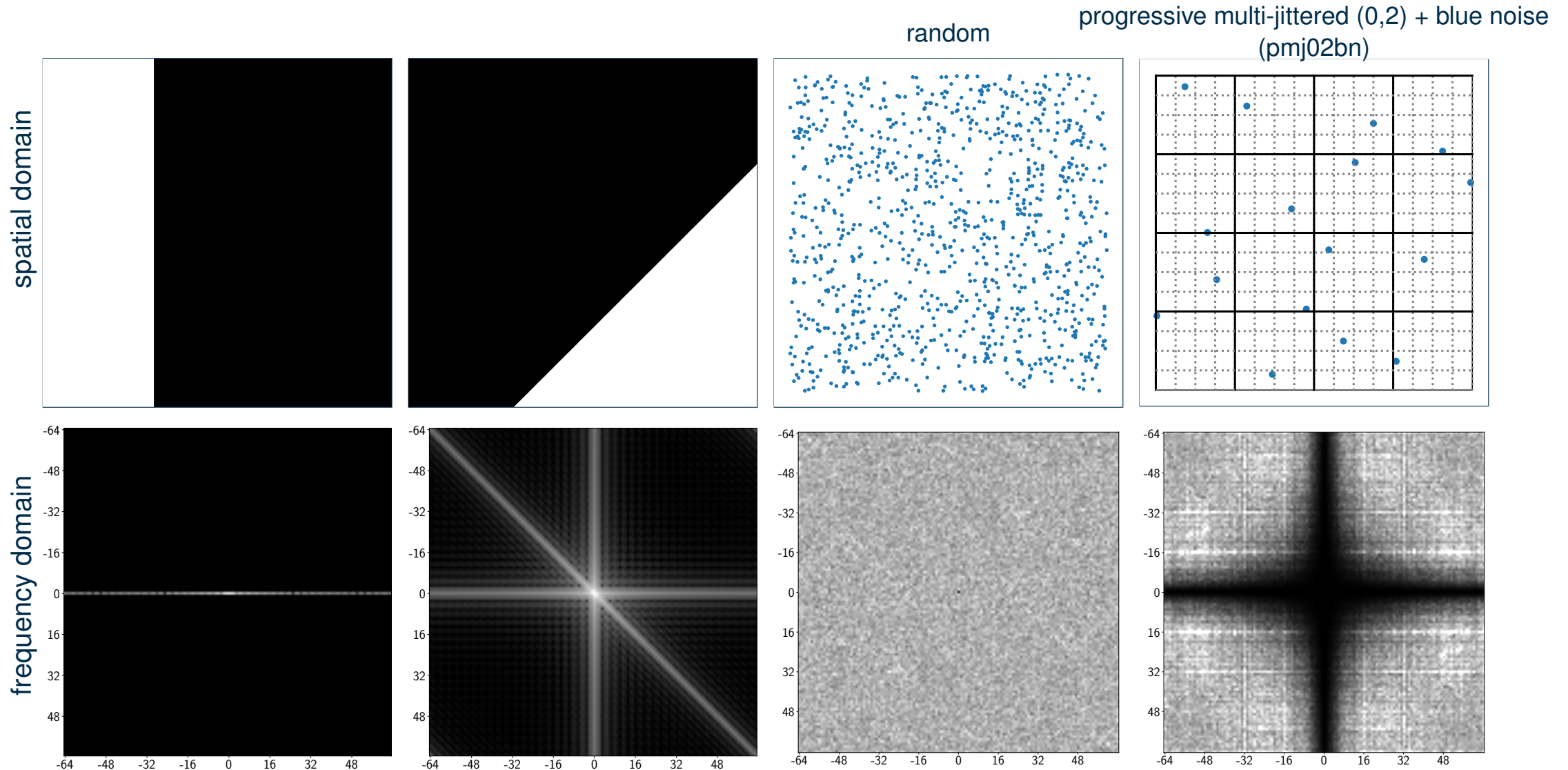
Discontinuous Integrands and Sampling Patterns



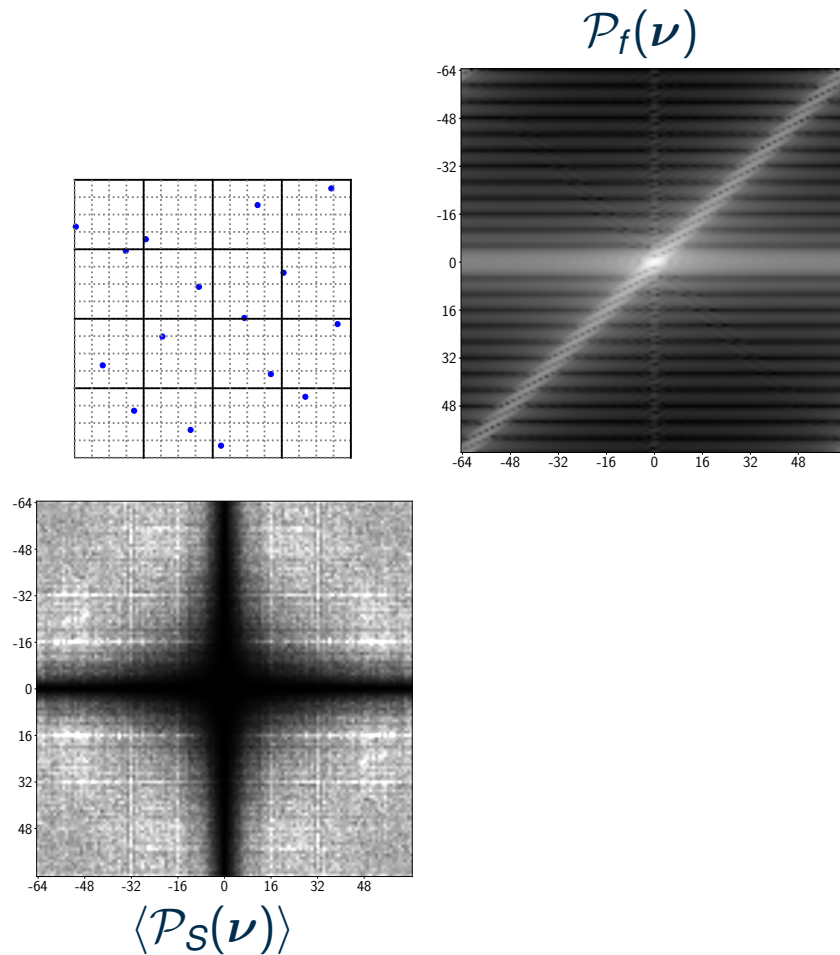
Discontinuous Integrands and Sampling Patterns



Discontinuous Integrands and Sampling Patterns

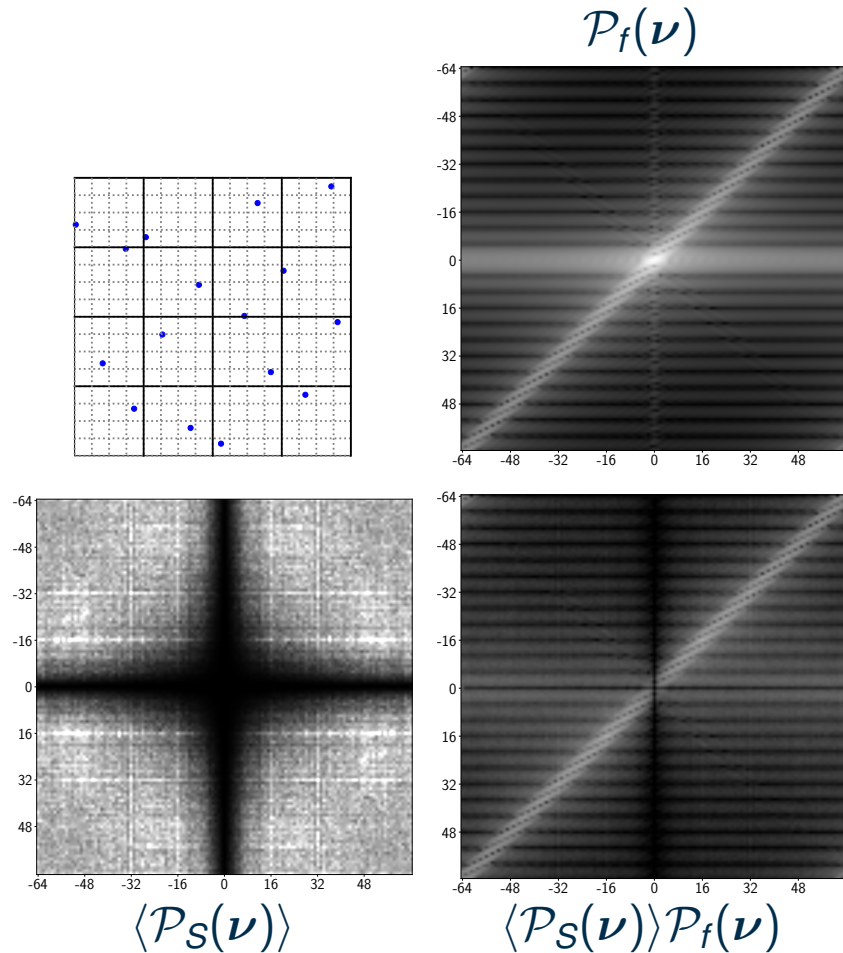


Aligning the Two Low-Energy Directions



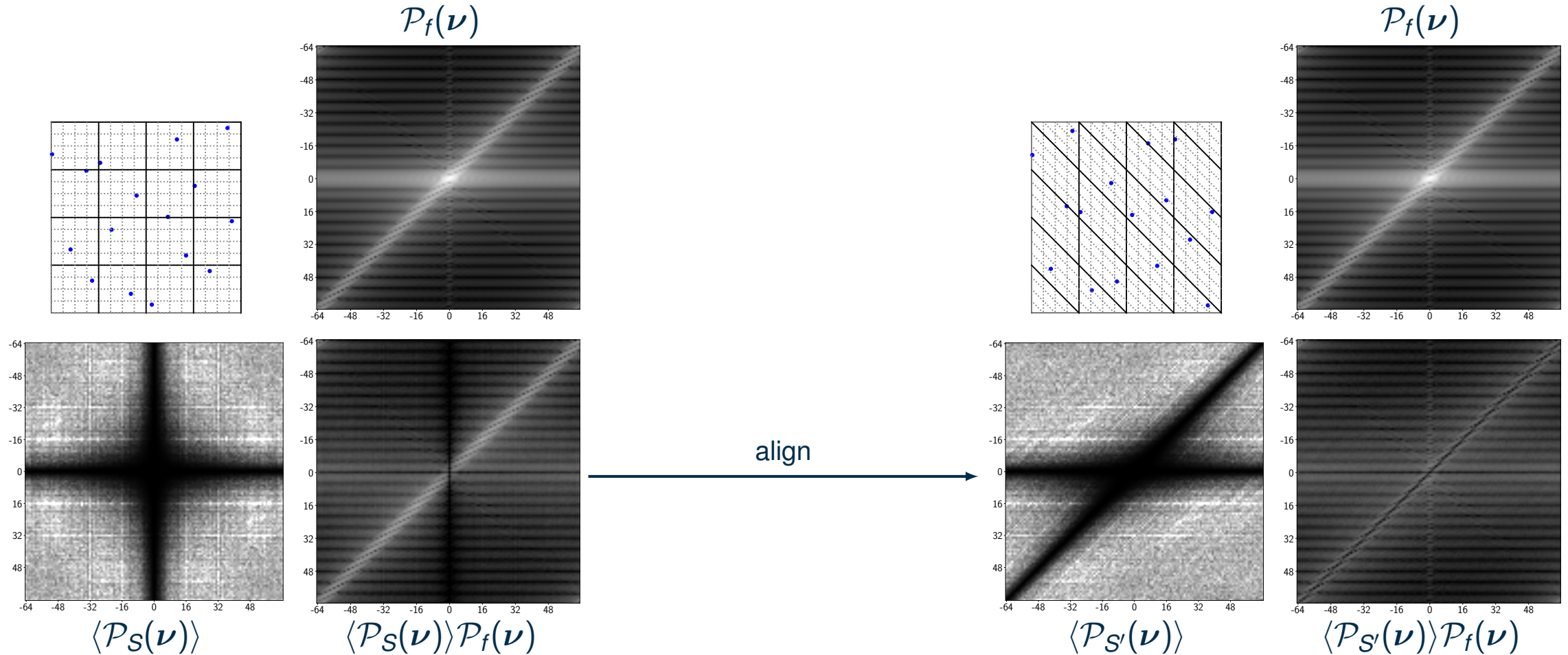
Convergence Analysis for Anisotropic Monte Carlo Sampling Spectra (Singh and Jarosz, 2017)

Aligning the Two Low-Energy Directions



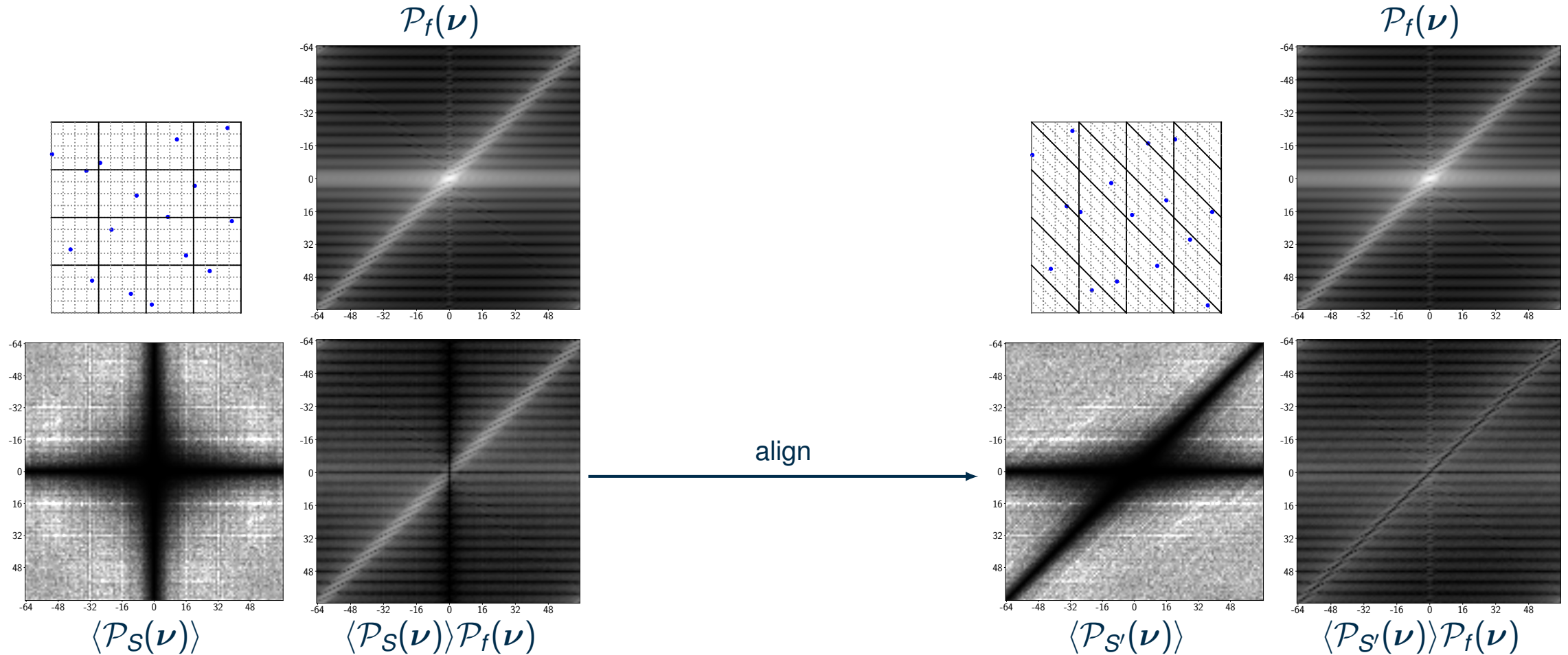
Convergence Analysis for Anisotropic Monte Carlo Sampling Spectra (Singh and Jarosz, 2017)

Aligning the Two Low-Energy Directions



Convergence Analysis for Anisotropic Monte Carlo Sampling Spectra (Singh and Jarosz, 2017)

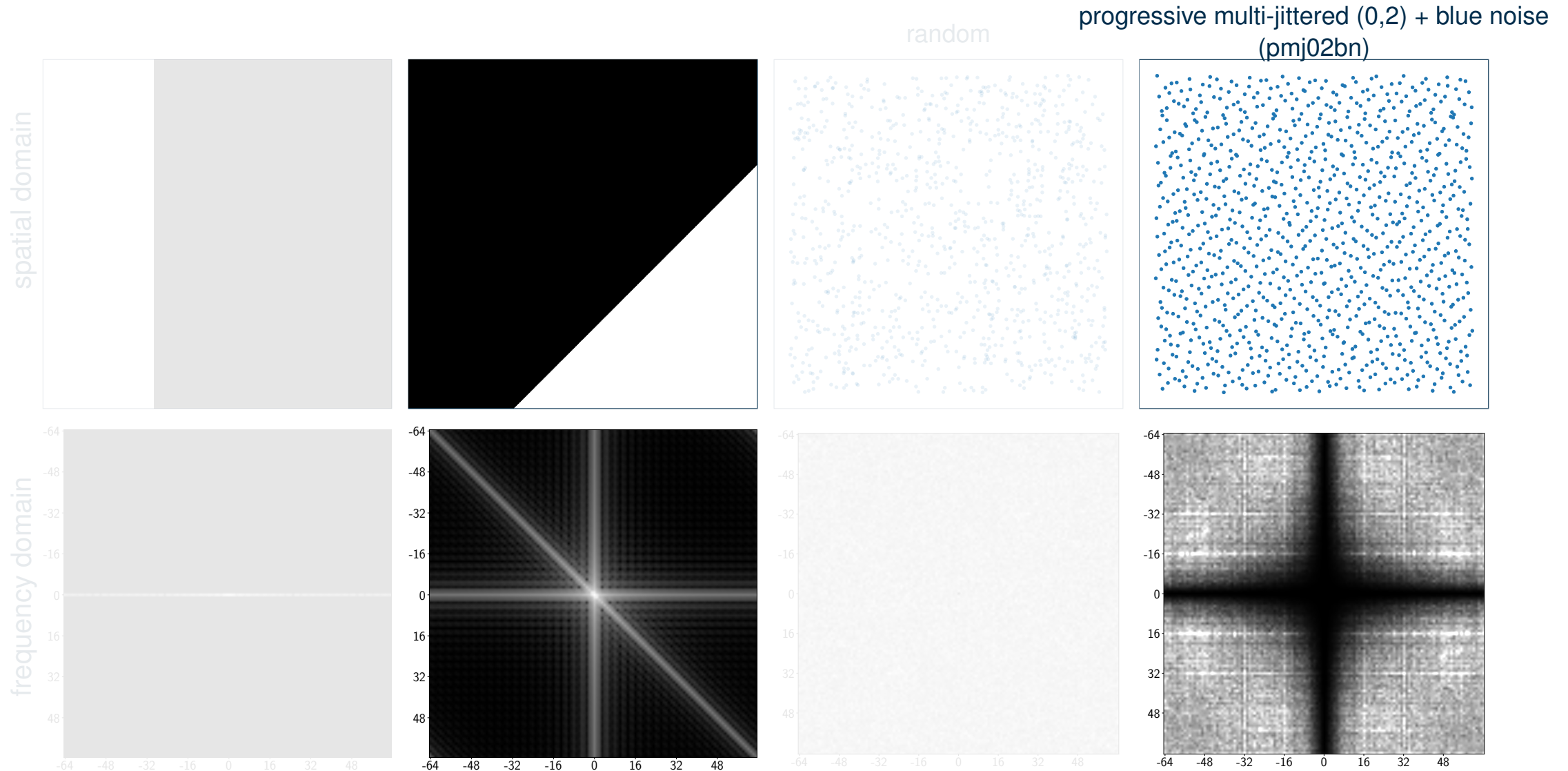
Aligning the Two Low-Energy Directions



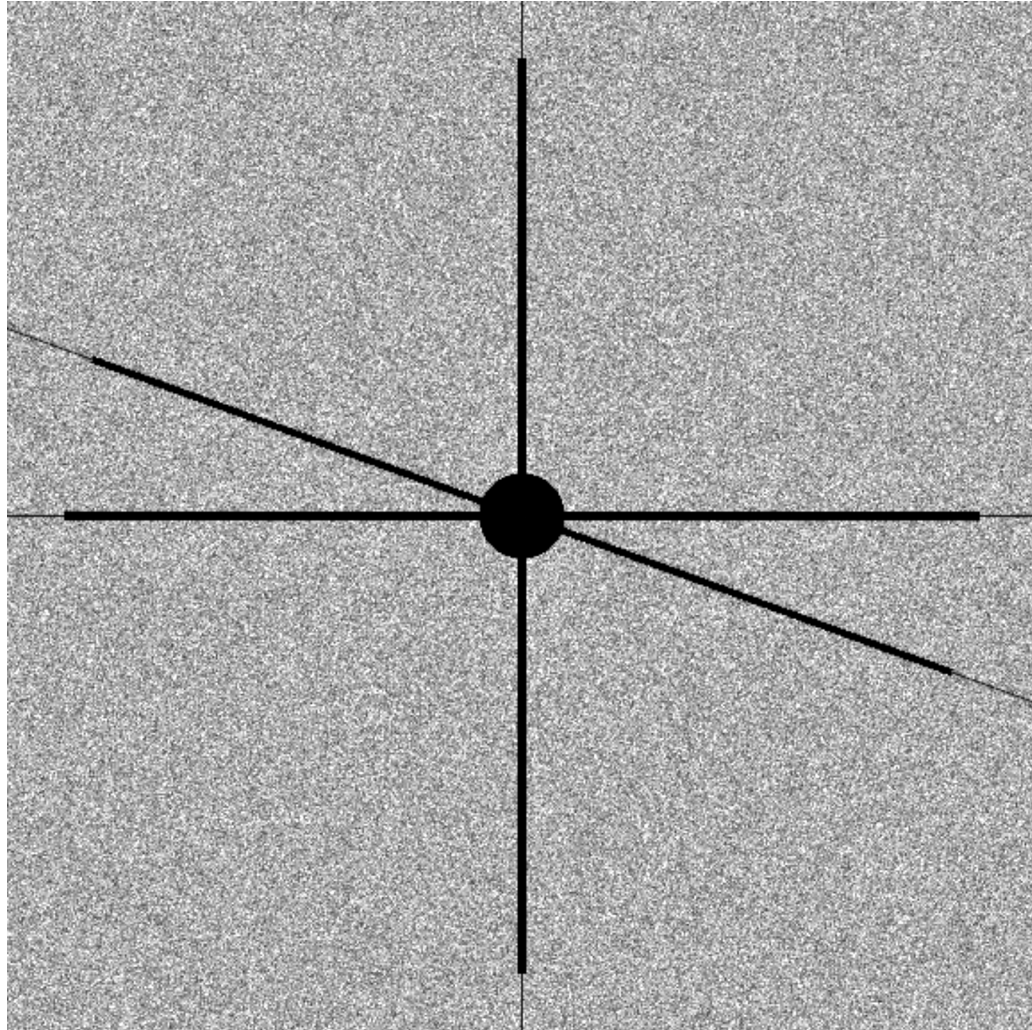
⇒ Existing anisotropic patterns only have two low-energy directions! ⚡

Convergence Analysis for Anisotropic Monte Carlo Sampling Spectra (Singh and Jarosz, 2017)

Discontinuous Integrands and Sampling Patterns



Optimization of Anisotropic Points

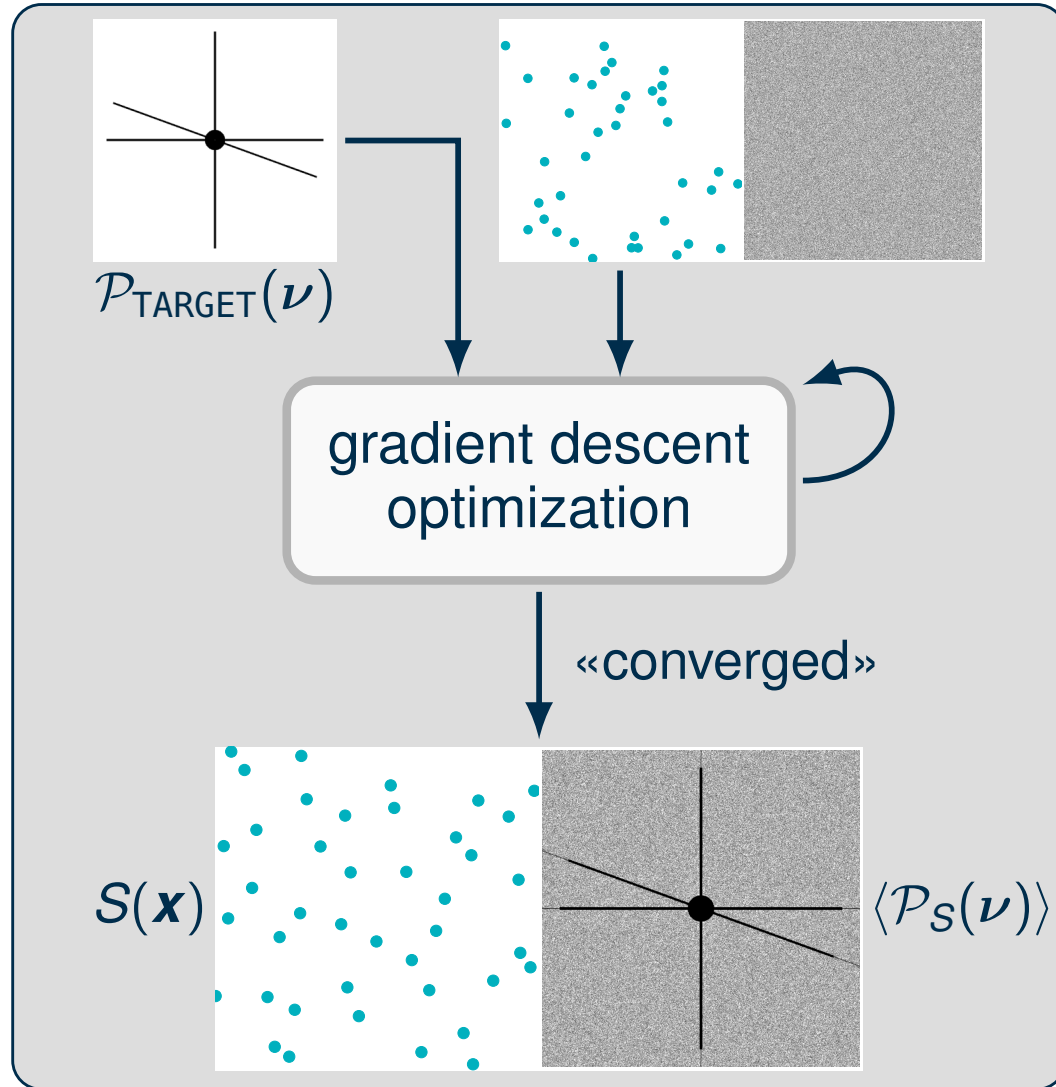


- goals

- suppress frequencies along multiple directions (stratification)
- suppress low frequencies (blue noise)

related work: Deep Point Correlation Design (Leimkühler et al., 2019), Sliced Optimal Transport Sampling (Paulin et al., 2020)

Optimization of Anisotropic Points



- goals

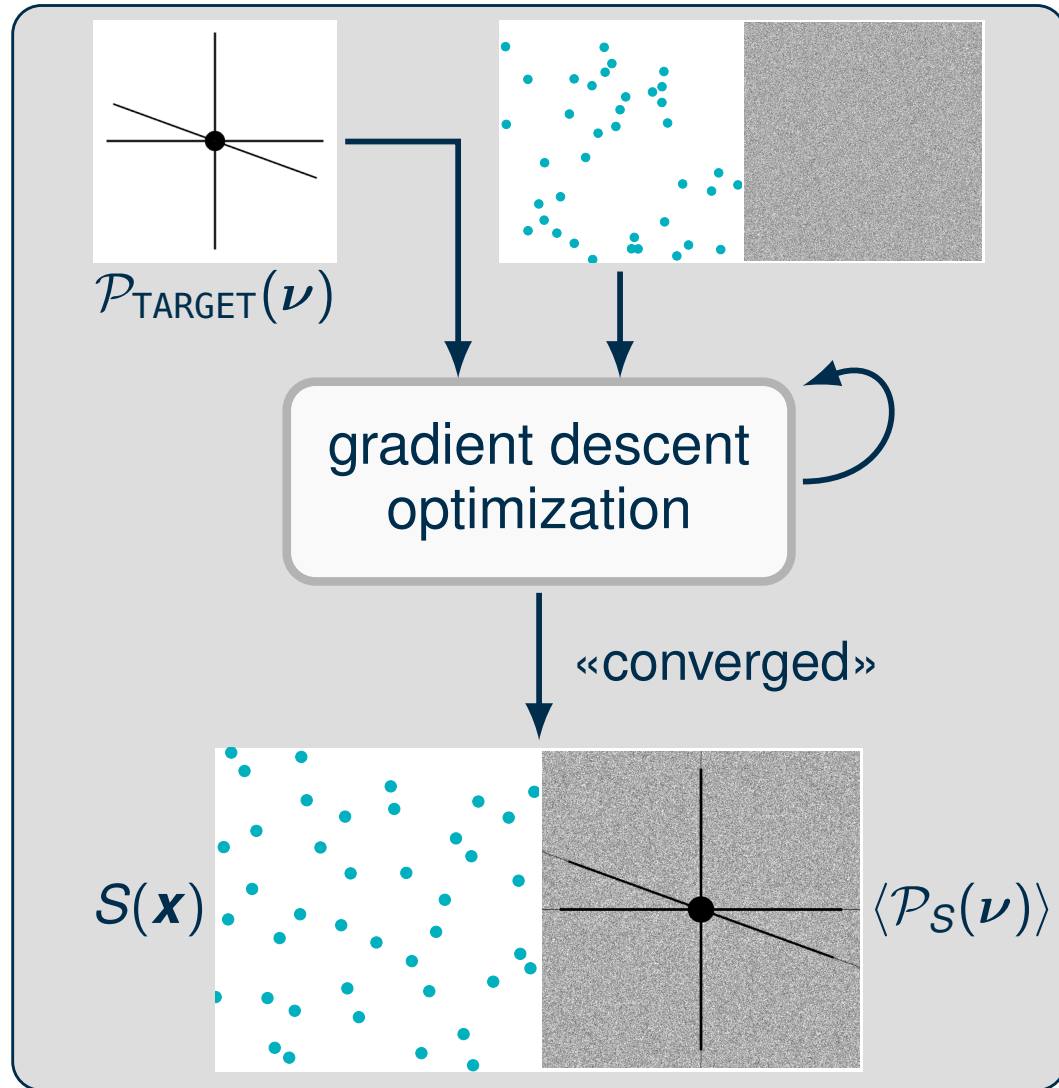
- suppress frequencies along multiple directions (stratification)
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- optimize locations of the points

$$\mathcal{L}(S) = \mathcal{L}_{\text{spec}}(S) + c\mathcal{L}_{\text{SW}}(S)$$

related work: Deep Point Correlation Design (Leimkühler et al., 2019), Sliced Optimal Transport Sampling (Paulin et al., 2020)

Optimization of Anisotropic Points



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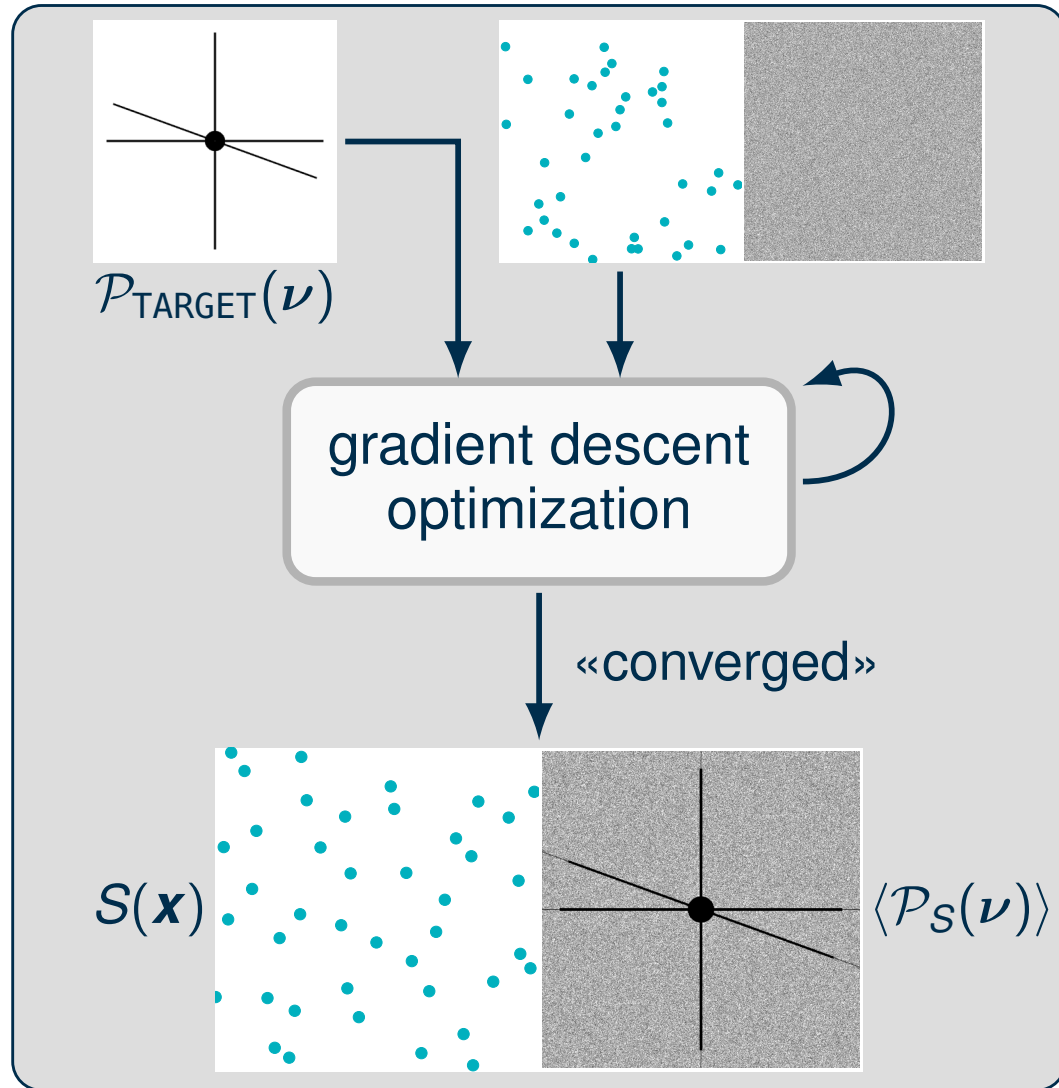
$$\mathcal{L}(S) = \mathcal{L}_{\text{spec}}(S) + c\mathcal{L}_{\text{SW}}(S)$$

- $\mathcal{L}_{\text{spec}}(S)$

- spectrum loss
- MSE between current $\mathcal{P}_S(\nu)$ and target $\mathcal{P}_{\text{TARGET}}(\nu)$ spectra

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- $\mathcal{L}_{\text{spec}}(S)$

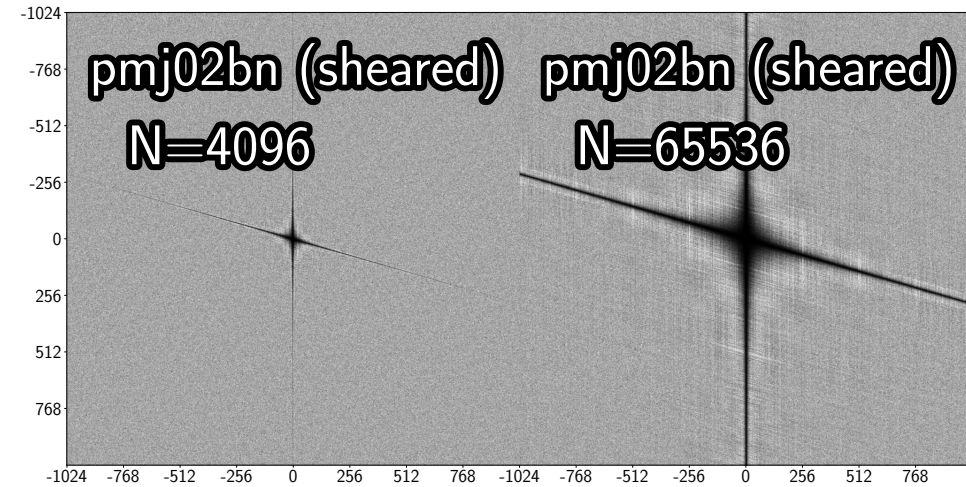
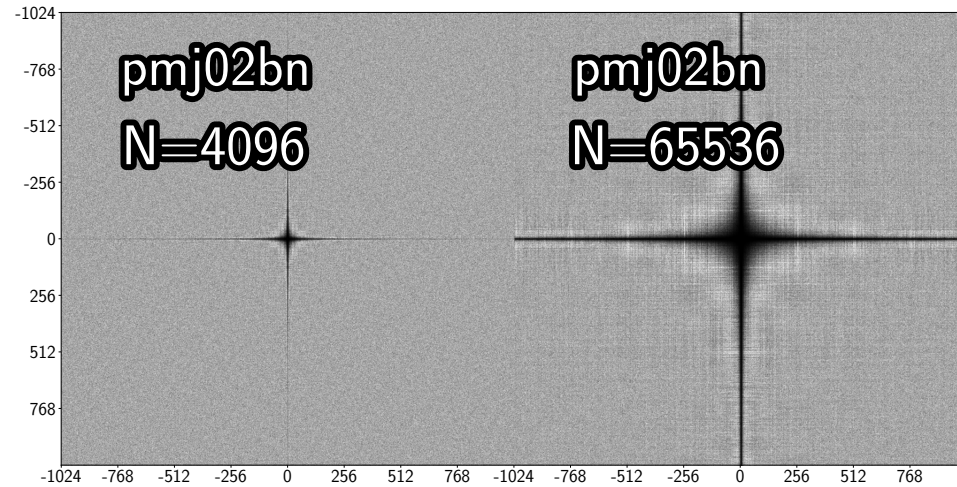
- spectrum loss
- MSE between current $\mathcal{P}_S(\nu)$ and target $\mathcal{P}_{\text{TARGET}}(\nu)$ spectra

- $\mathcal{L}_{\text{SW}}(S)$

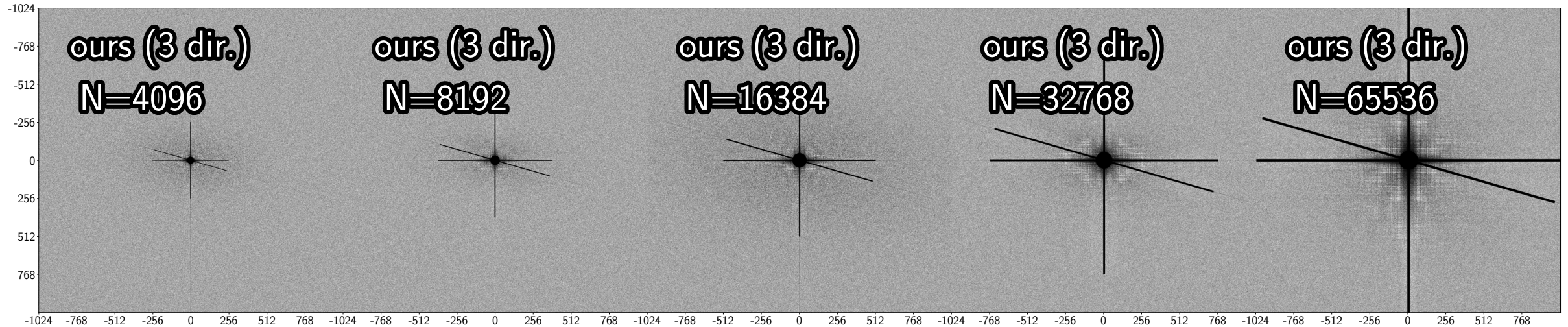
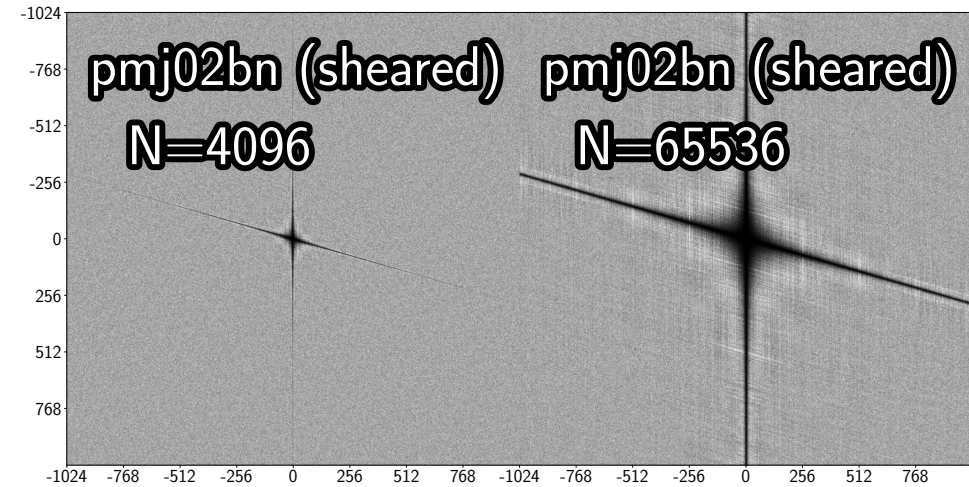
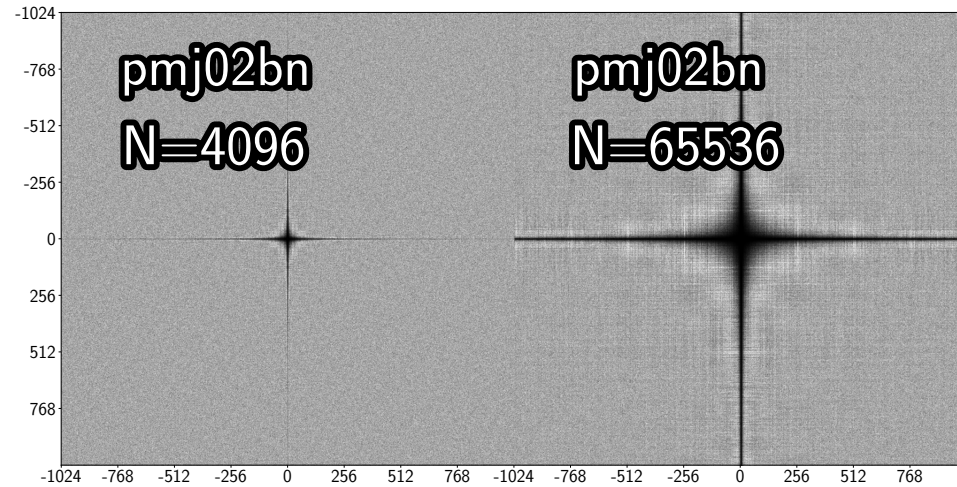
- sliced Wasserstein loss (optimal transport)
- MSE between current and target point locations in 1D projections

related work: Deep Point Correlation Design (Leimkühler et al., 2019), Sliced Optimal Transport Sampling (Paulin et al., 2020)

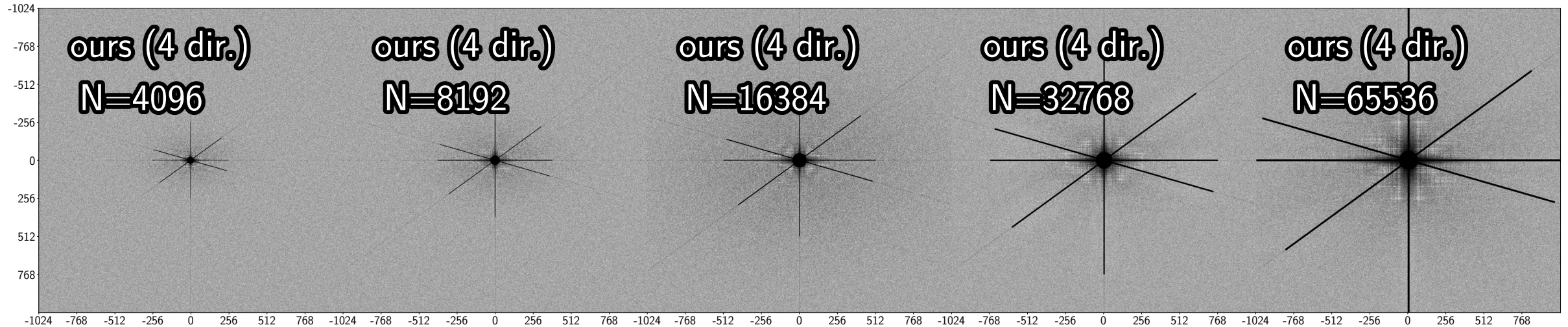
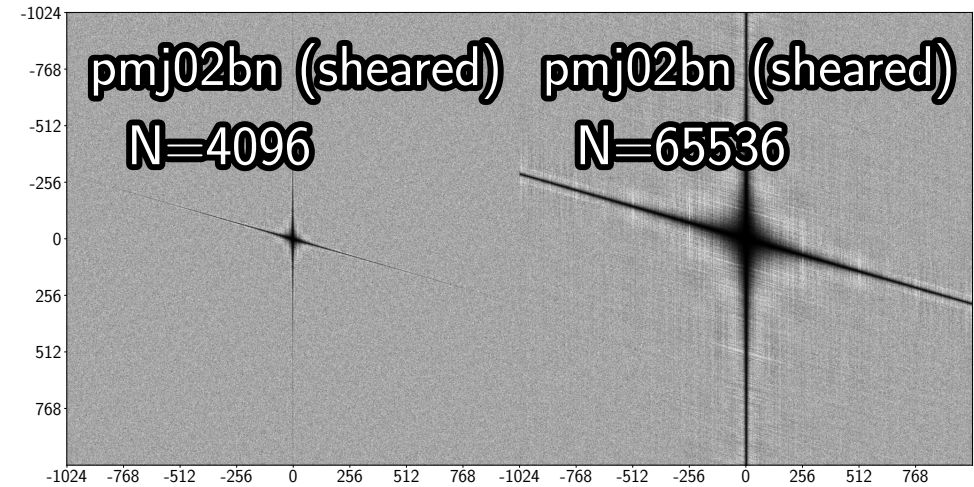
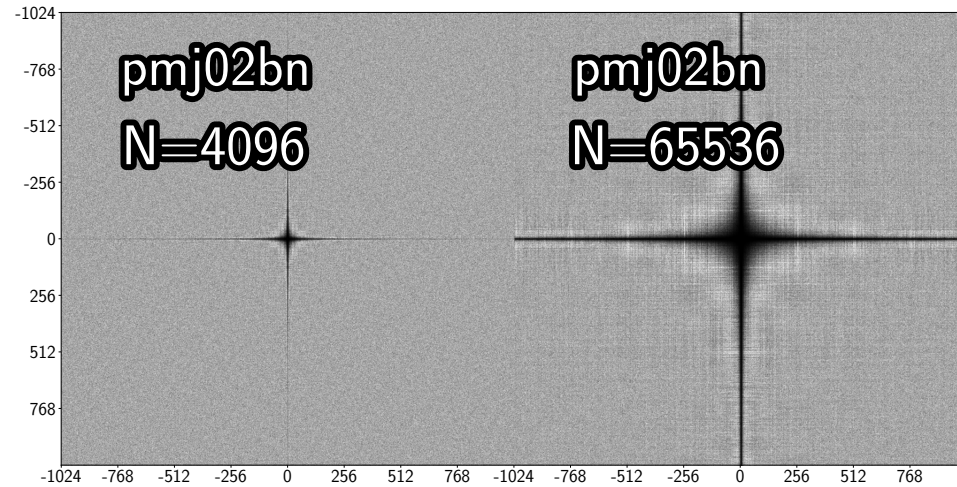
Optimization of N Anisotropic Points



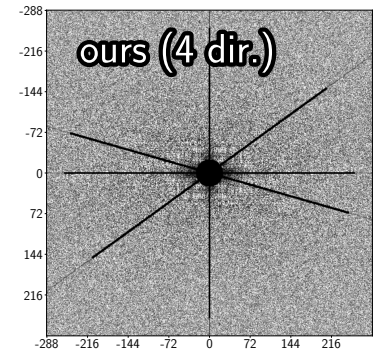
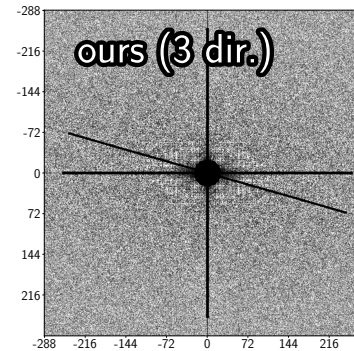
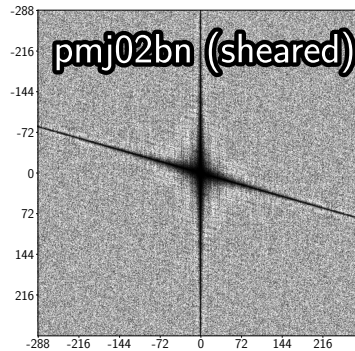
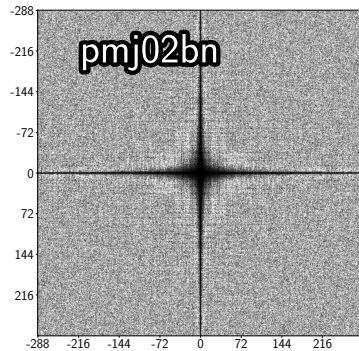
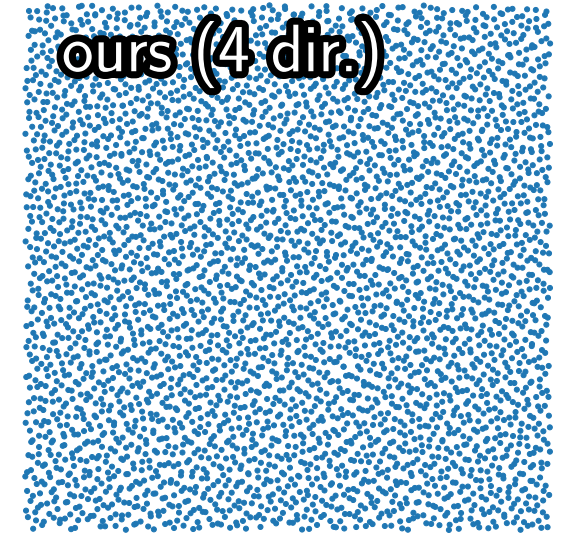
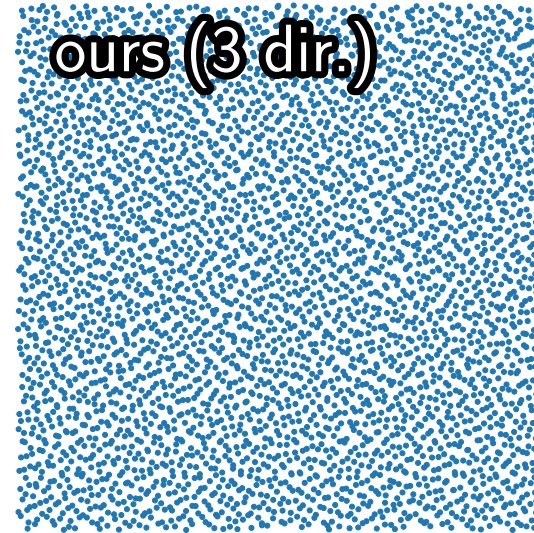
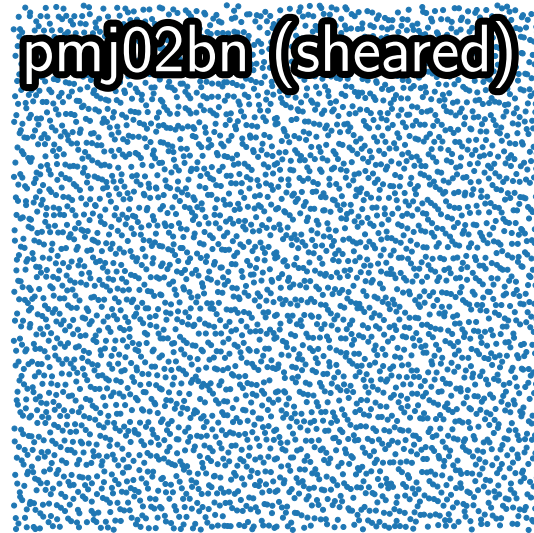
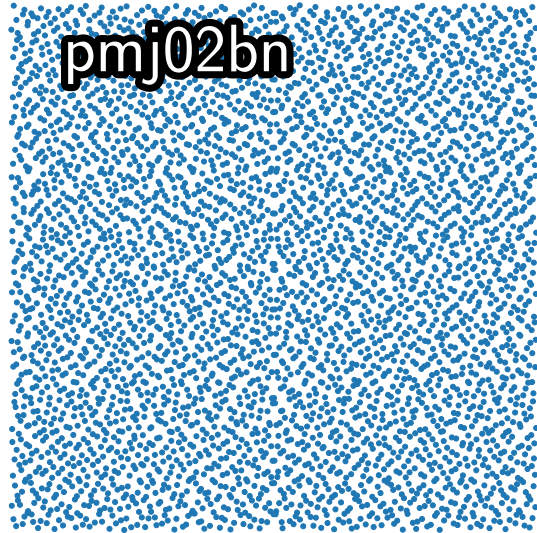
Optimization of N Anisotropic Points



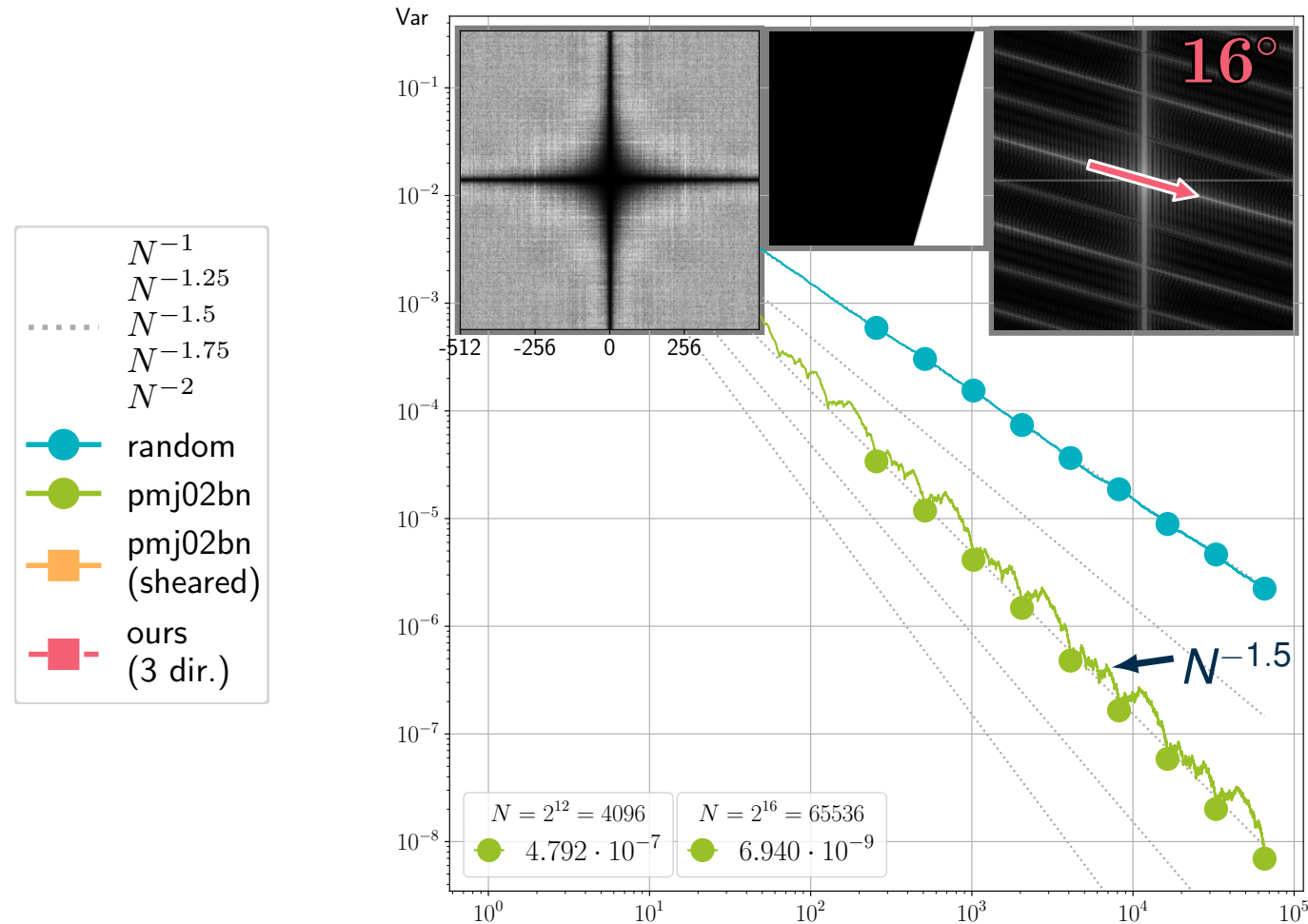
Optimization of N Anisotropic Points



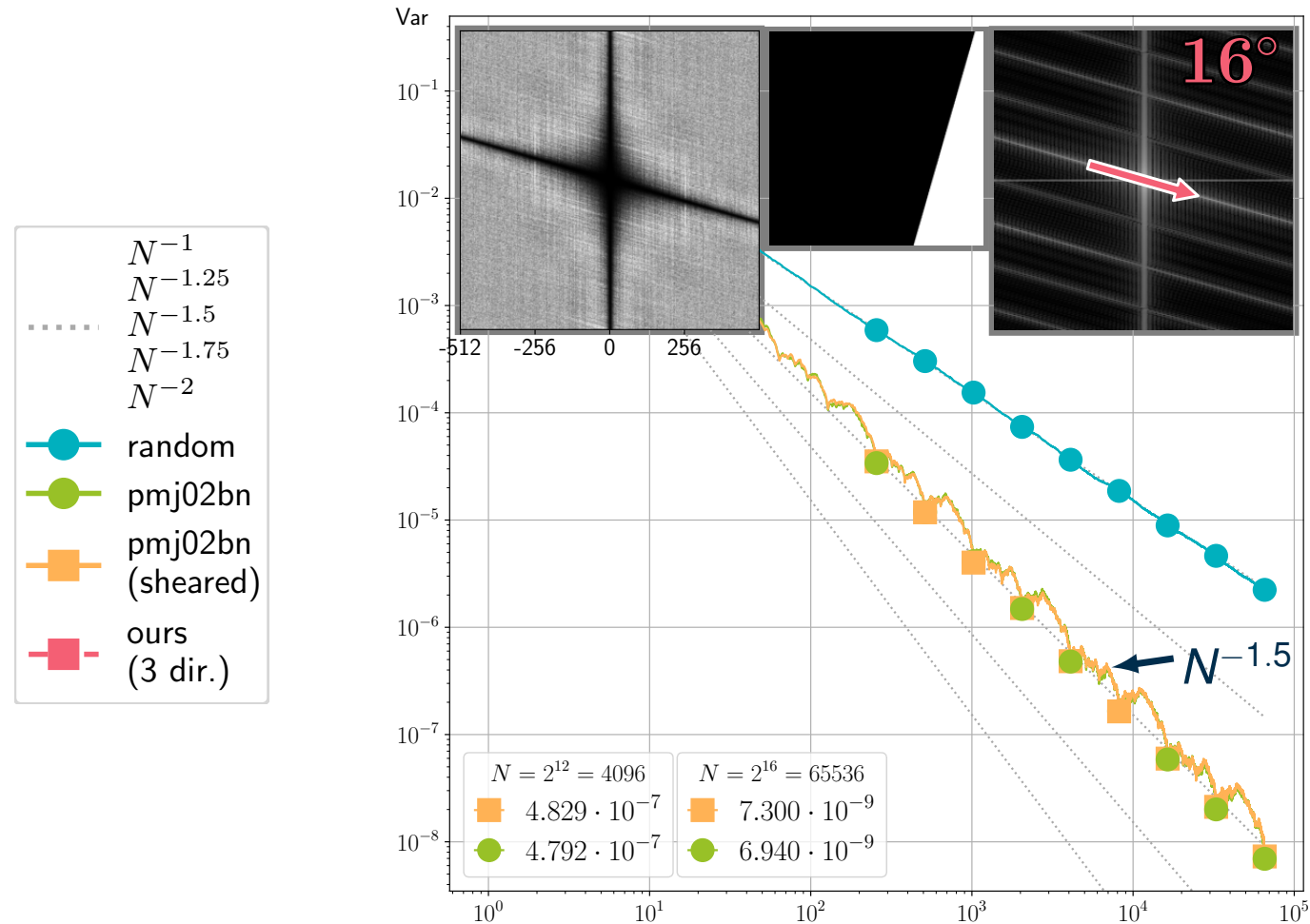
Optimization of N=4096 Anisotropic Points



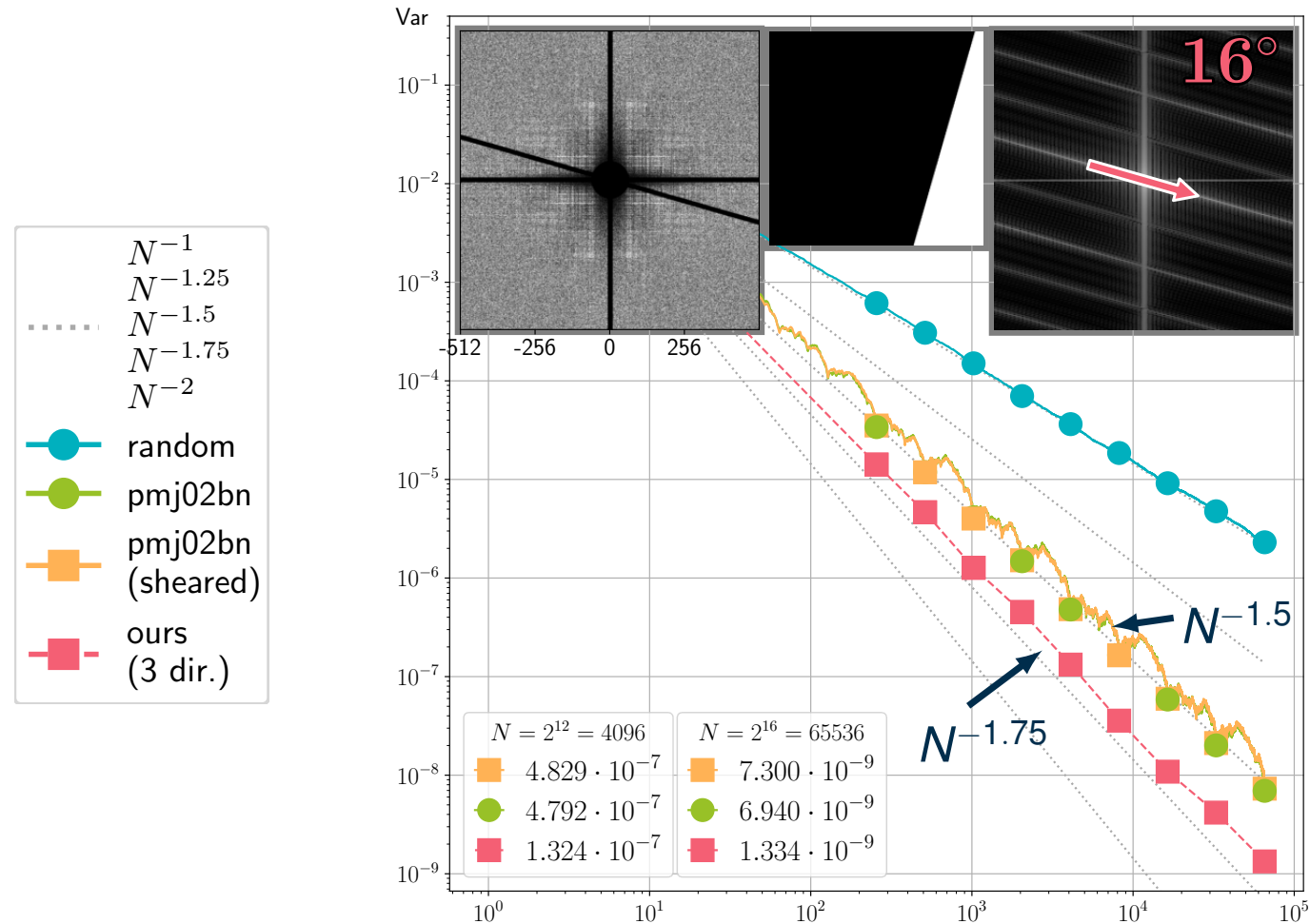
Optimization of Anisotropic Points: Results (3 dir.)



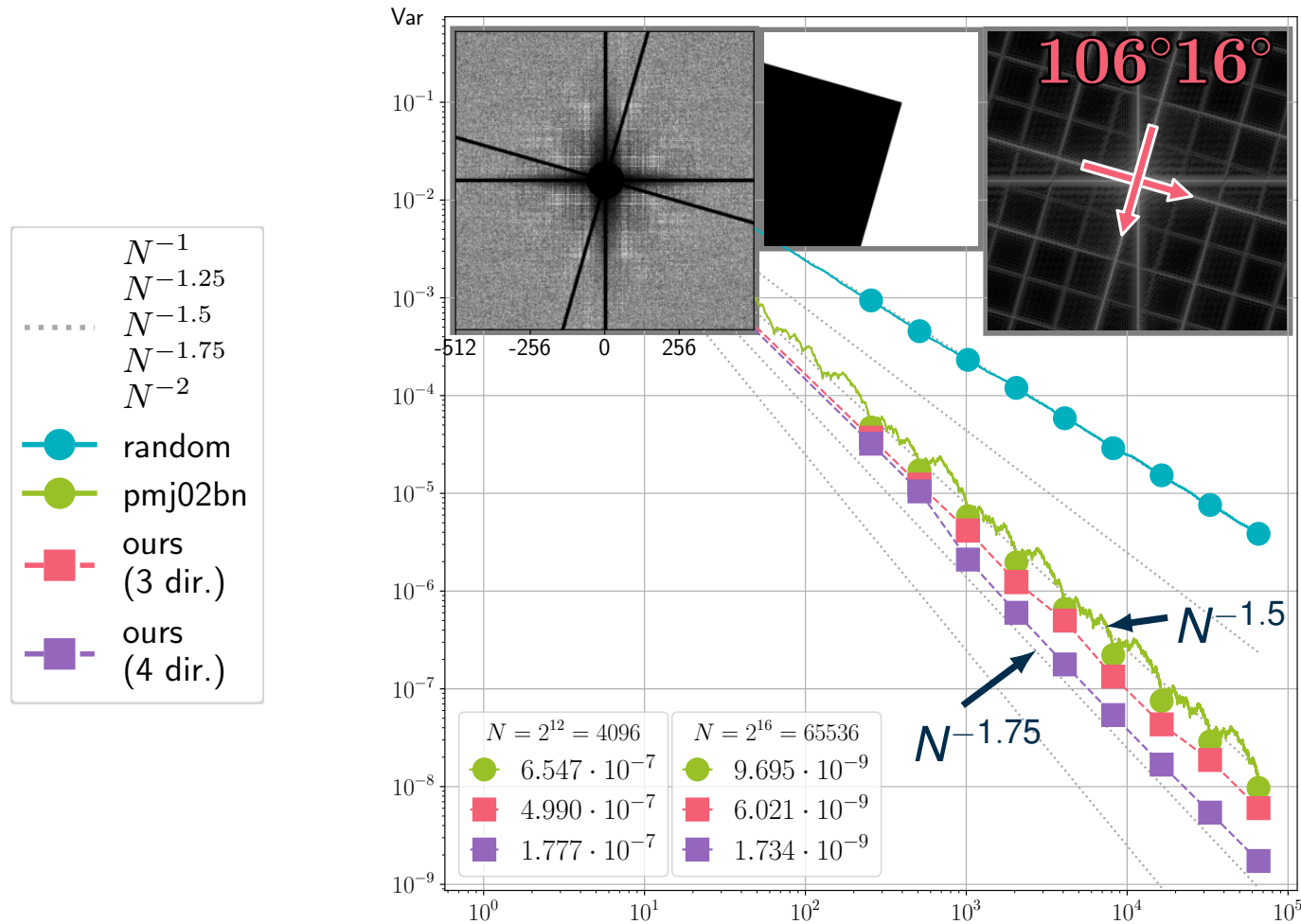
Optimization of Anisotropic Points: Results (3 dir.)



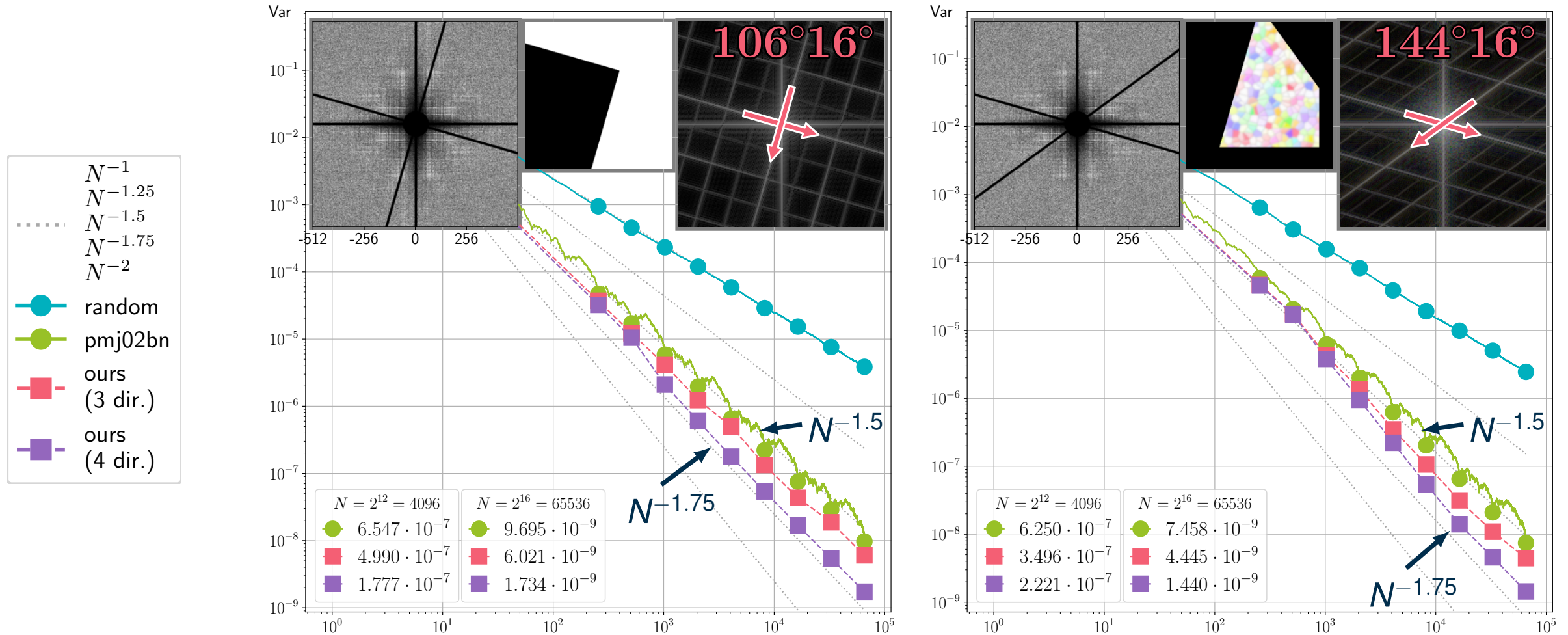
Optimization of Anisotropic Points: Results (3 dir.)



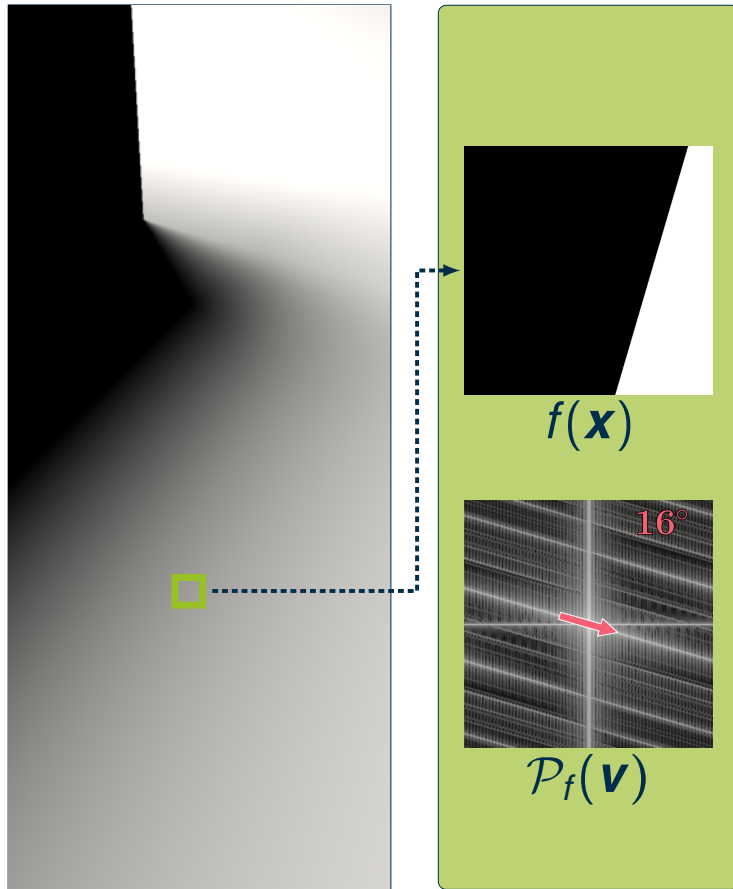
Optimization of Anisotropic Points: Results (4 dir.)



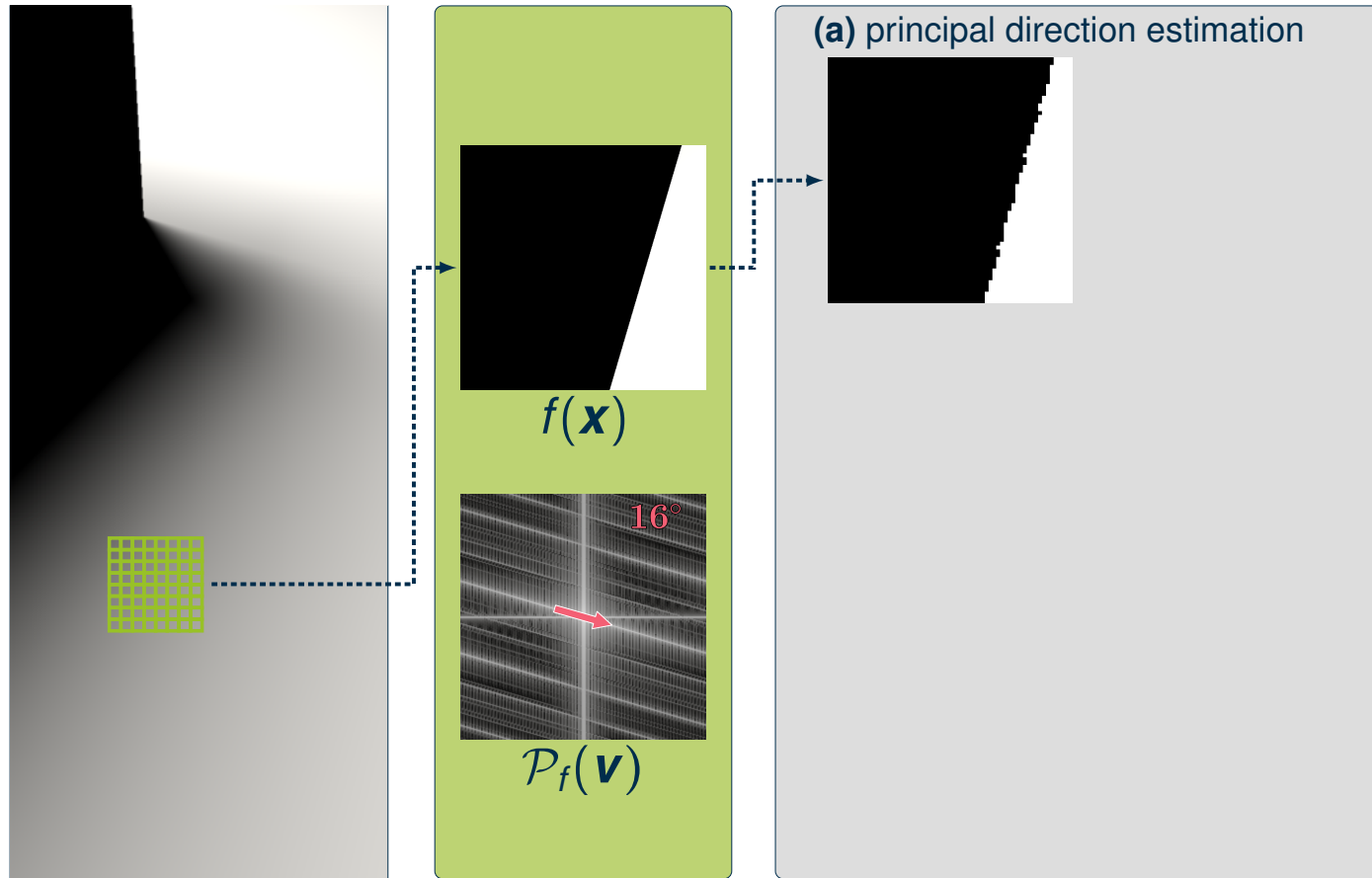
Optimization of Anisotropic Points: Results (4 dir.)



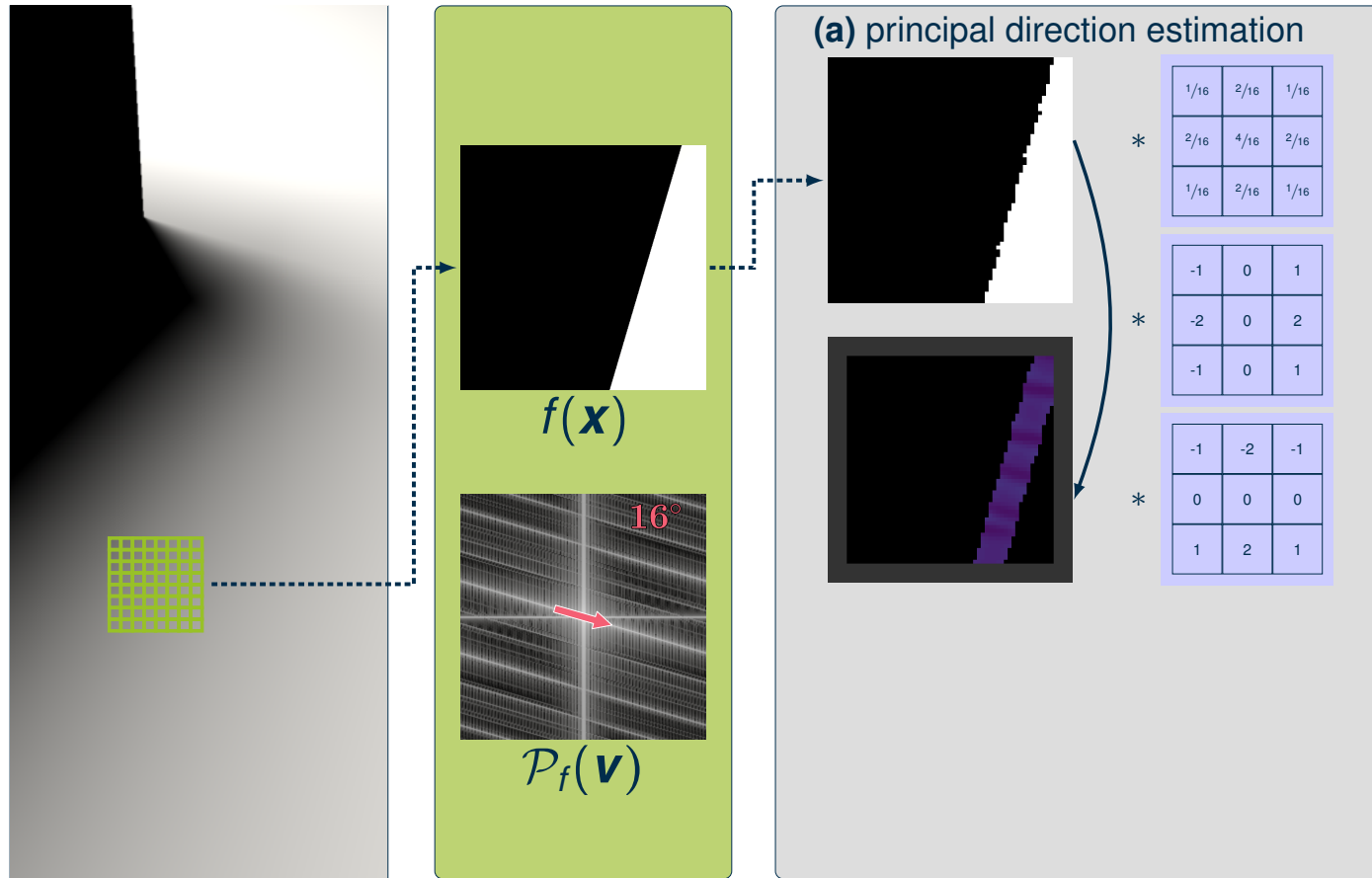
Offline Path Tracing With Area Light Sampling



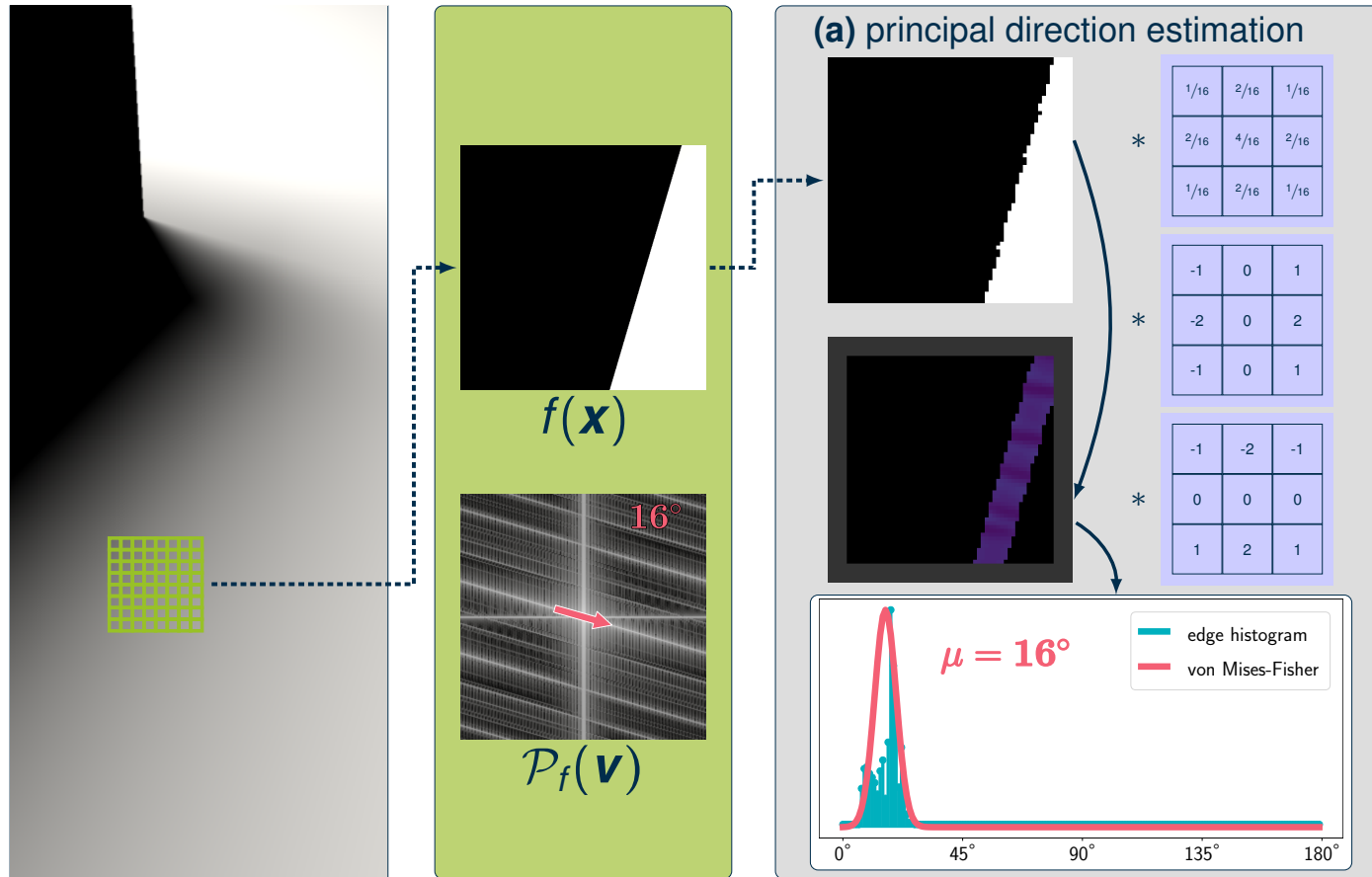
Offline Path Tracing With Area Light Sampling



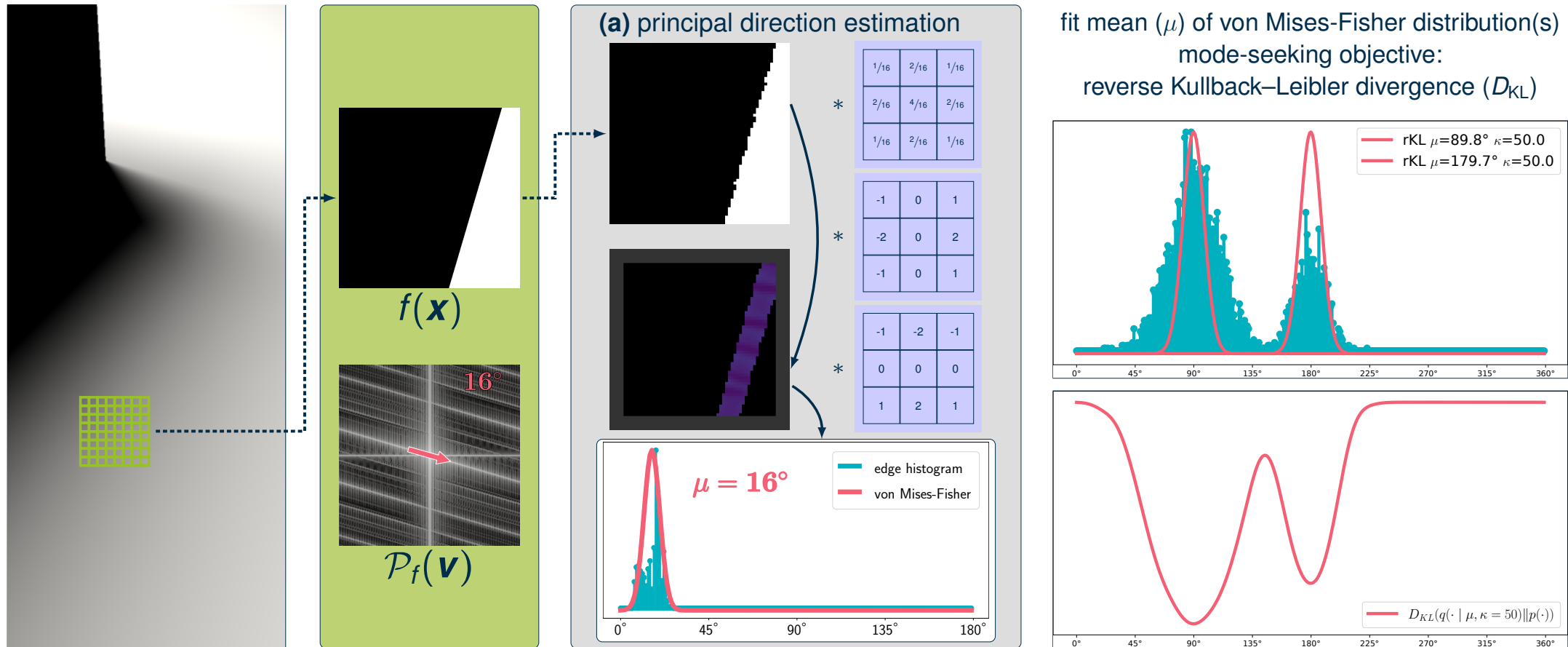
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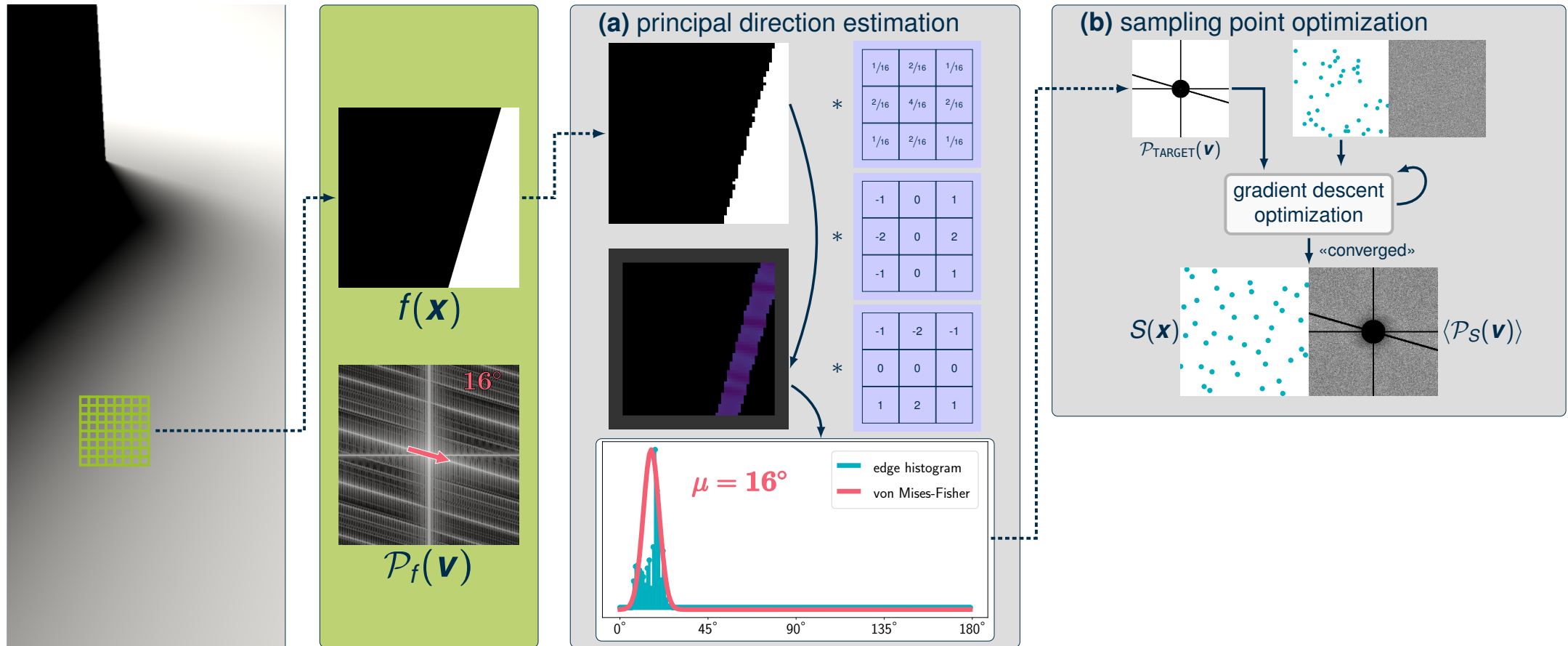


Offline Path Tracing With Area Light Sampling

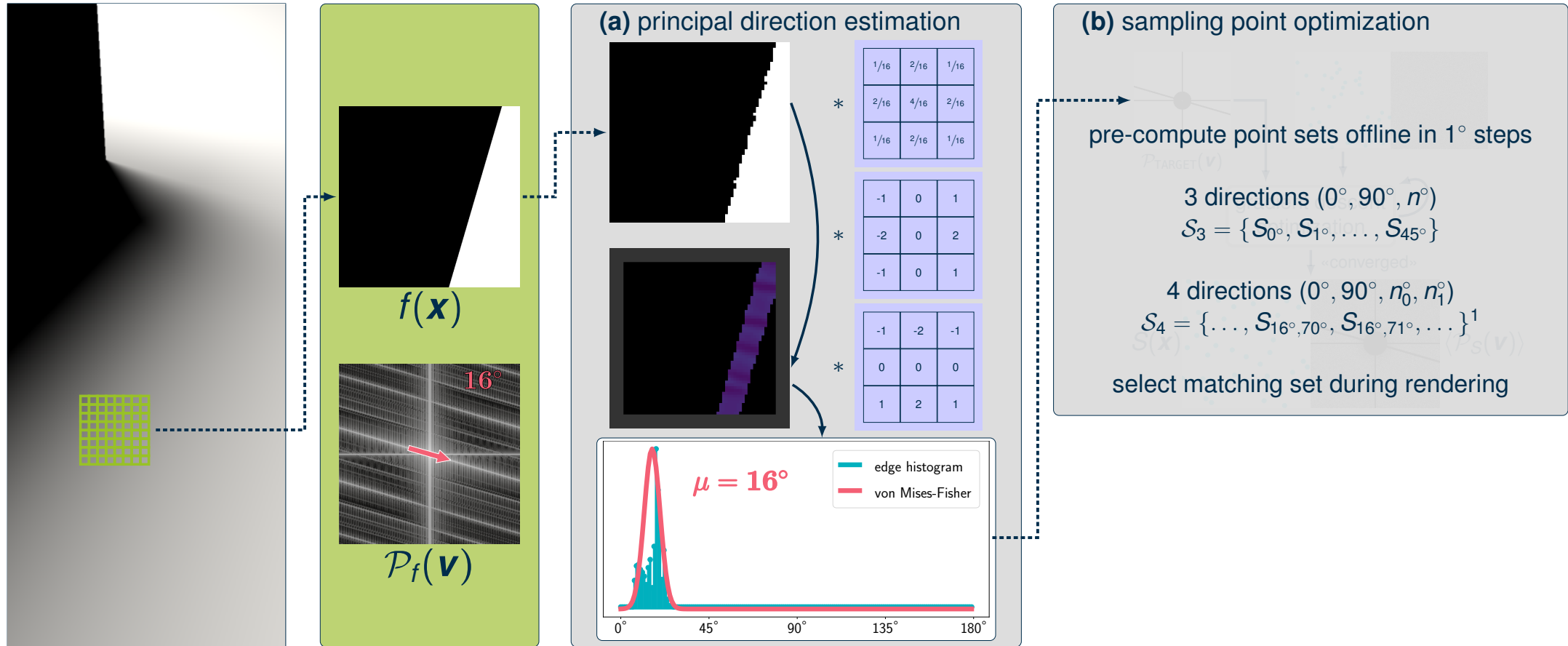


Pattern Recognition and Machine Learning p. 469 (Bishop, 2006), Kullback-Leibler Divergence (Kullback et al., 1951)

Offline Path Tracing With Area Light Sampling

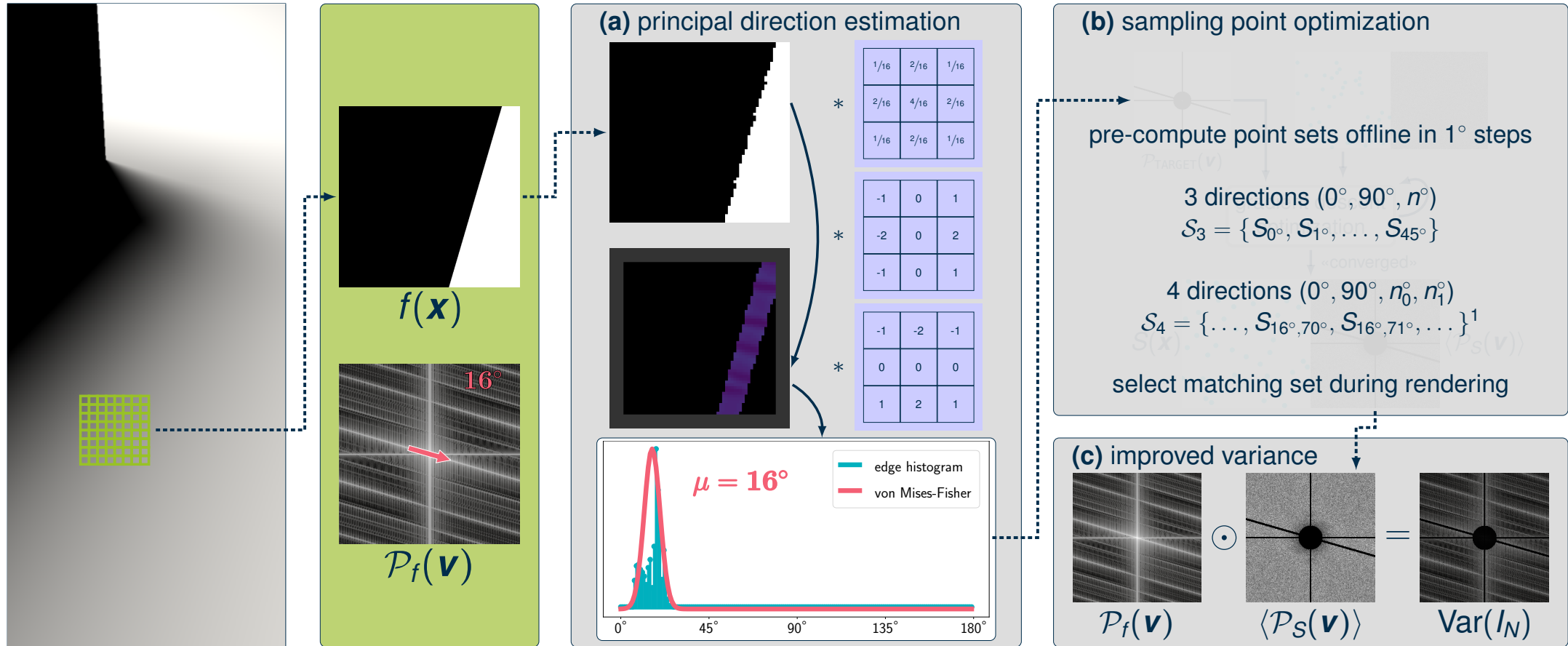


Offline Path Tracing With Area Light Sampling



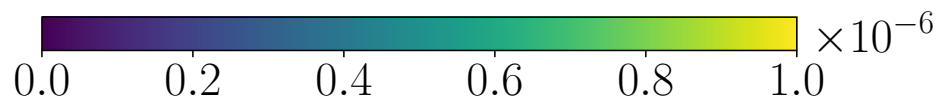
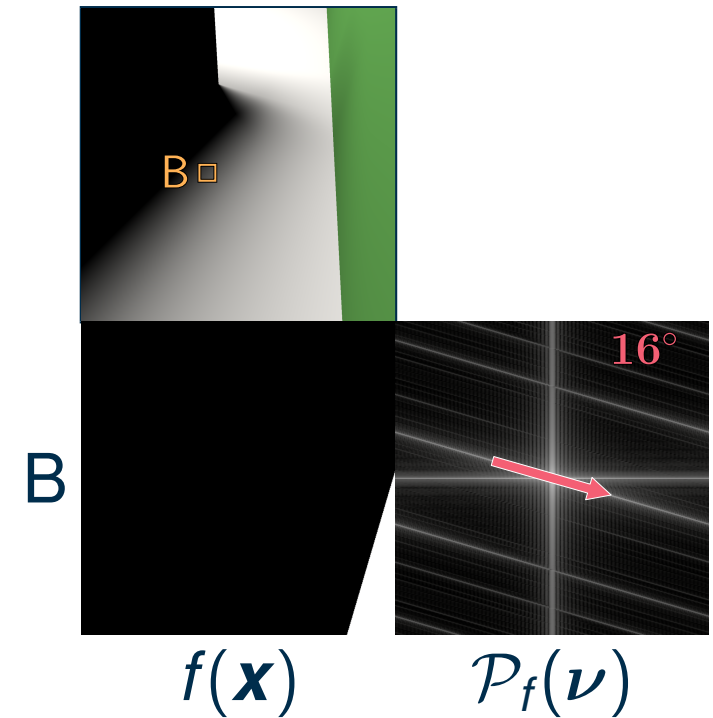
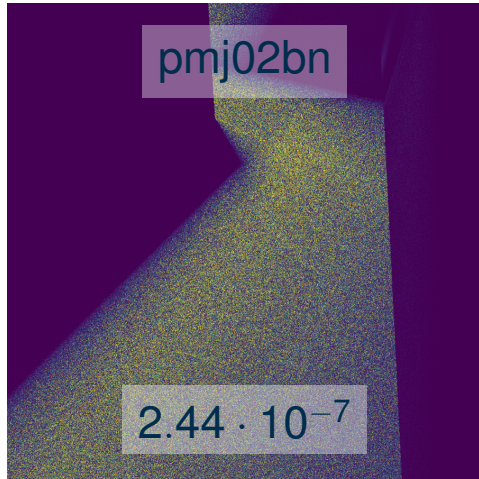
¹ $|S_4| = 4141$; requires about 136 MB in total for 2^{12} points per set

Offline Path Tracing With Area Light Sampling



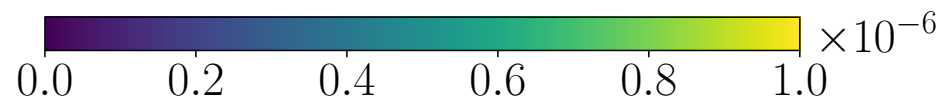
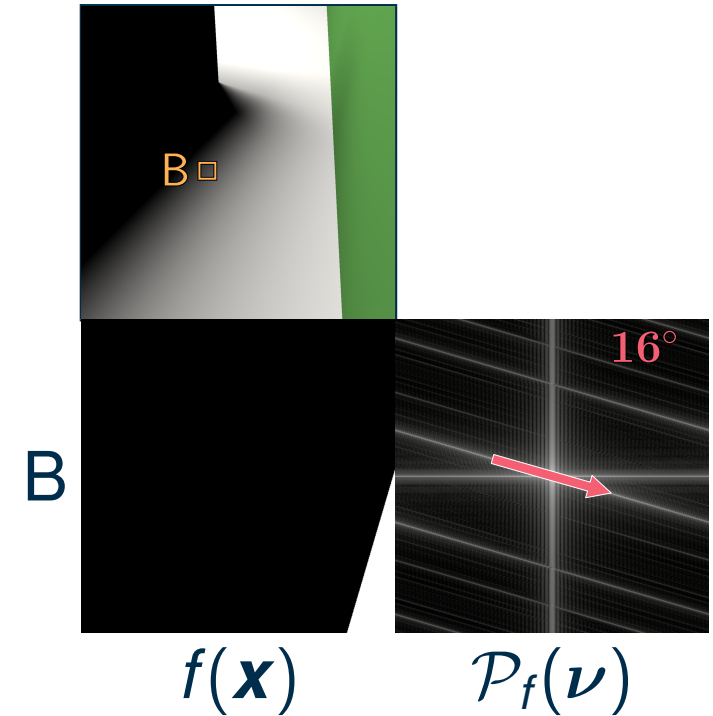
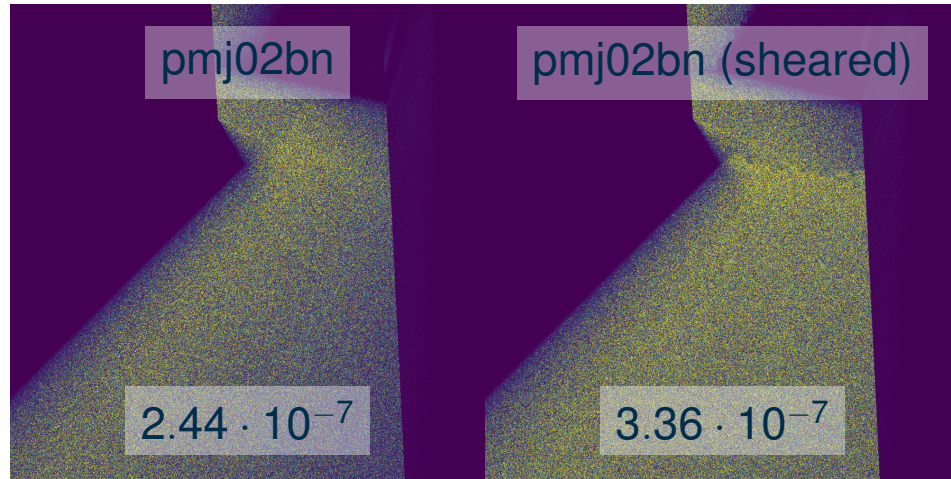
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Results: $\text{Var}(I_N)$, $N=4096$ spp, Area Light Sampling



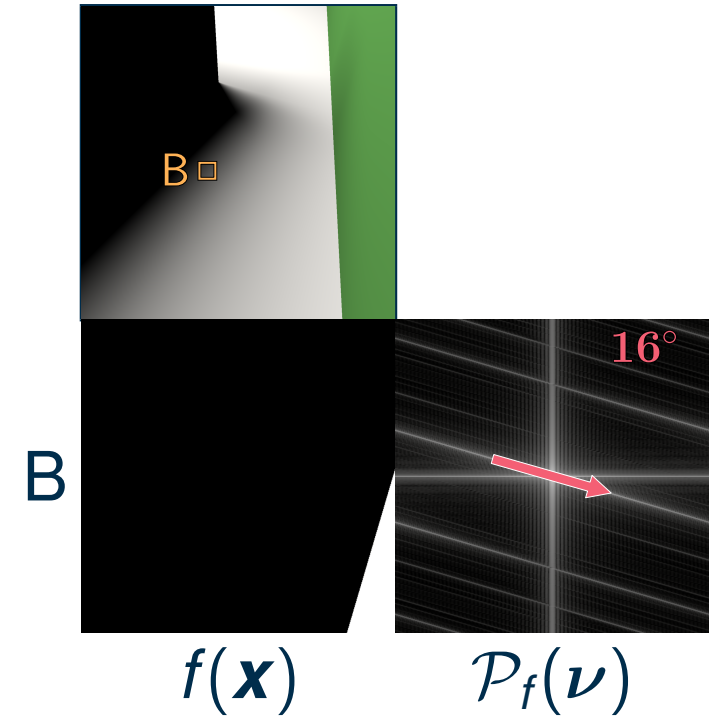
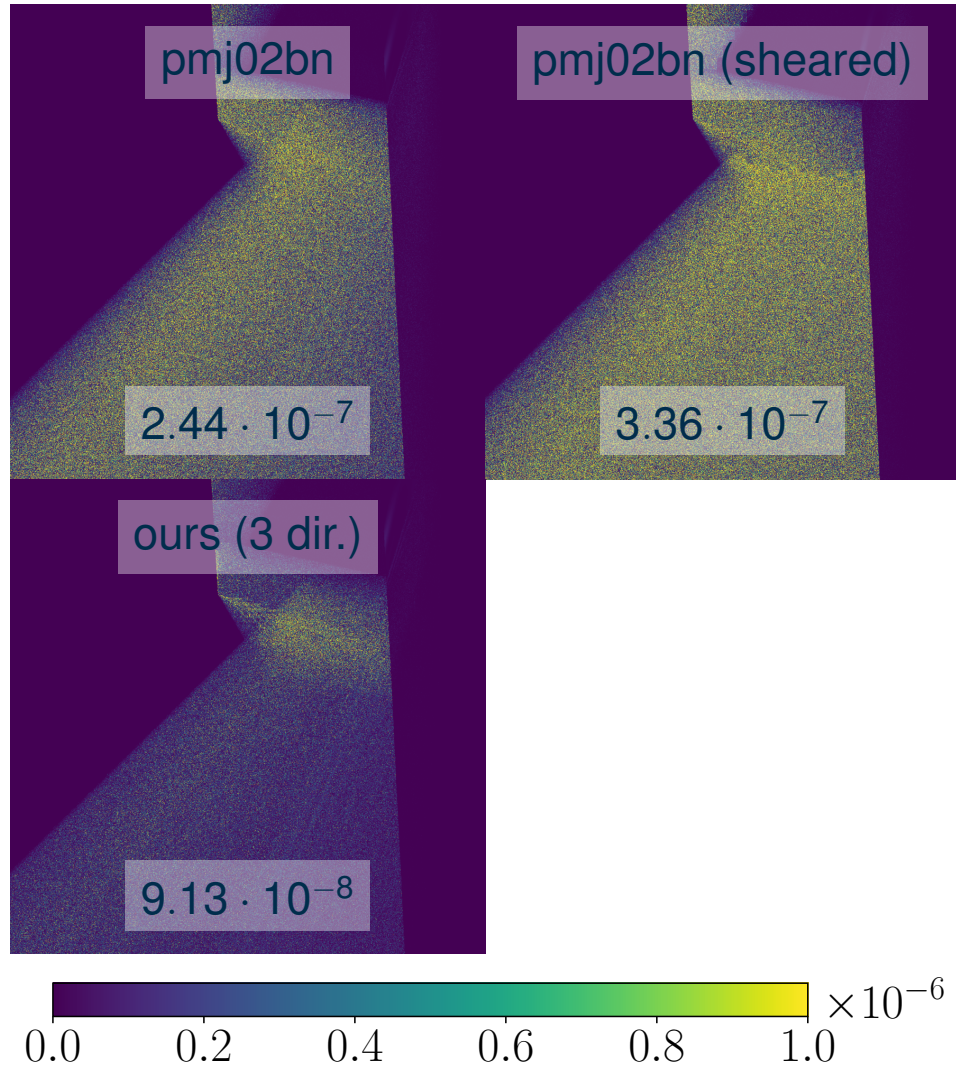
results for other scenes in the paper

Results: $\text{Var}(I_N)$, $N=4096$ spp, Area Light Sampling



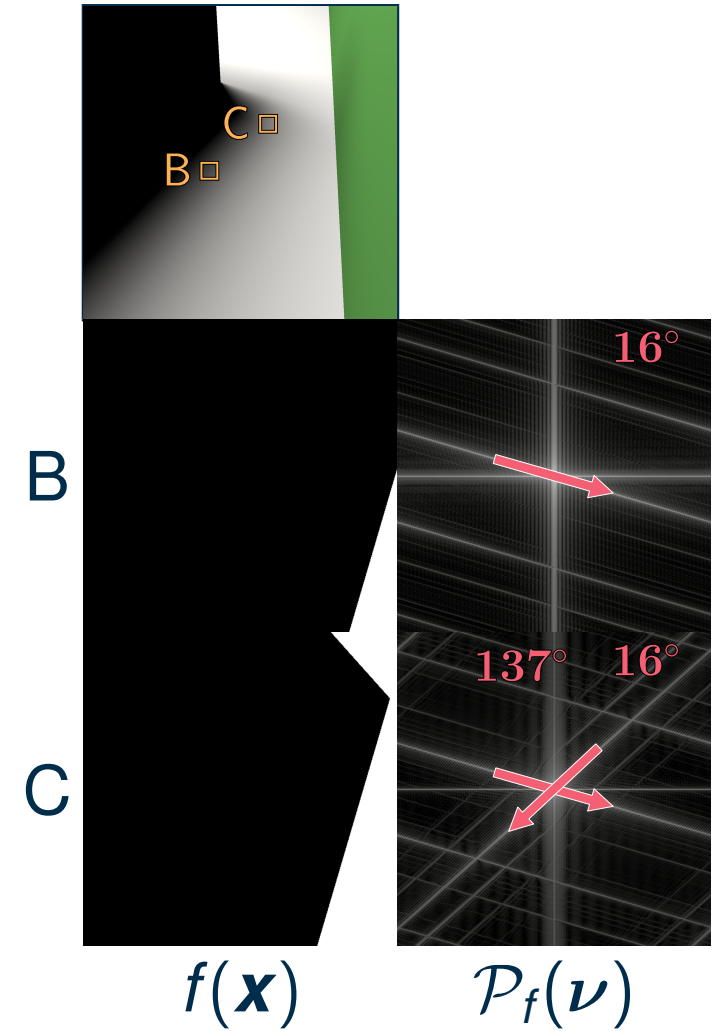
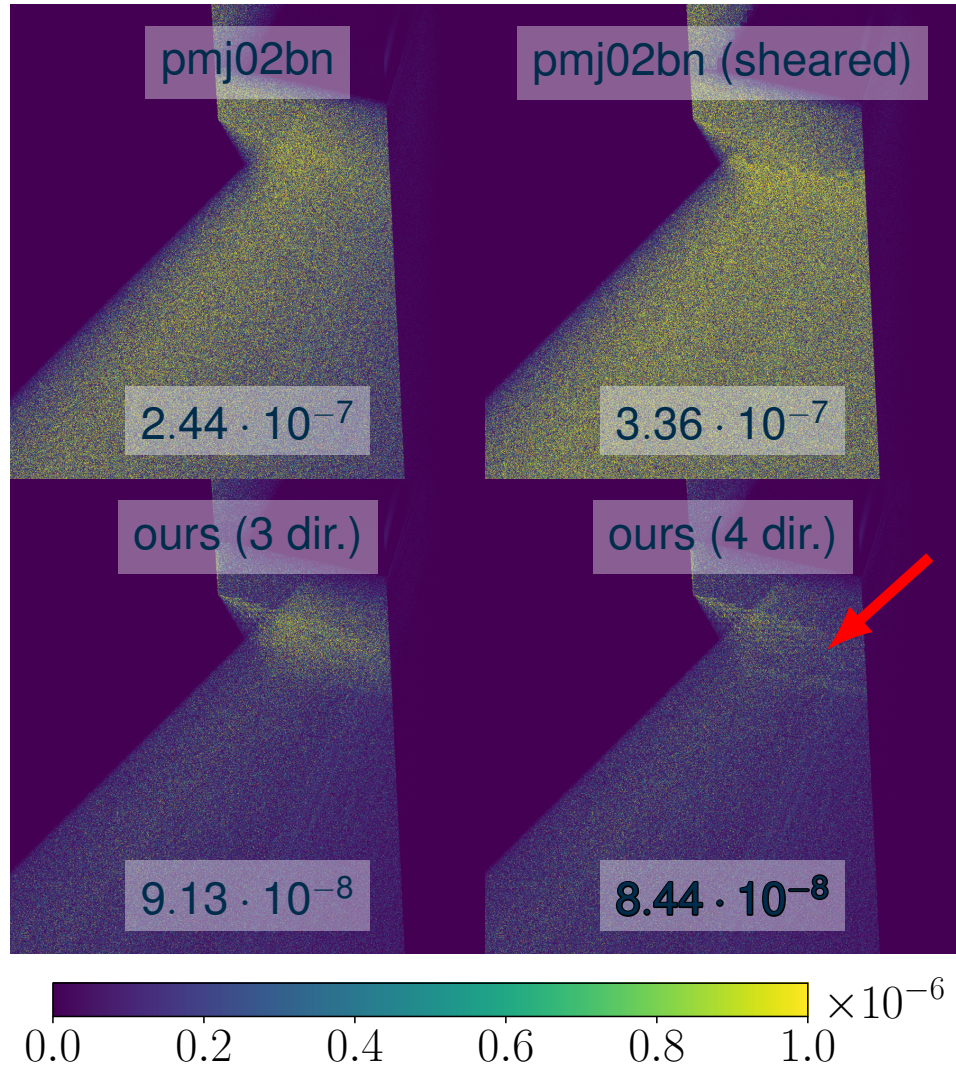
results for other scenes in the paper

Results: $\text{Var}(I_N)$, $N=4096$ spp, Area Light Sampling



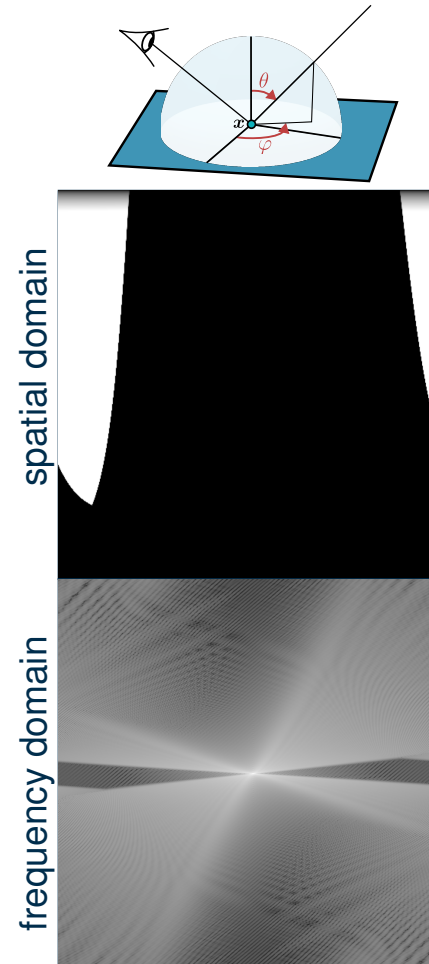
results for other scenes in the paper

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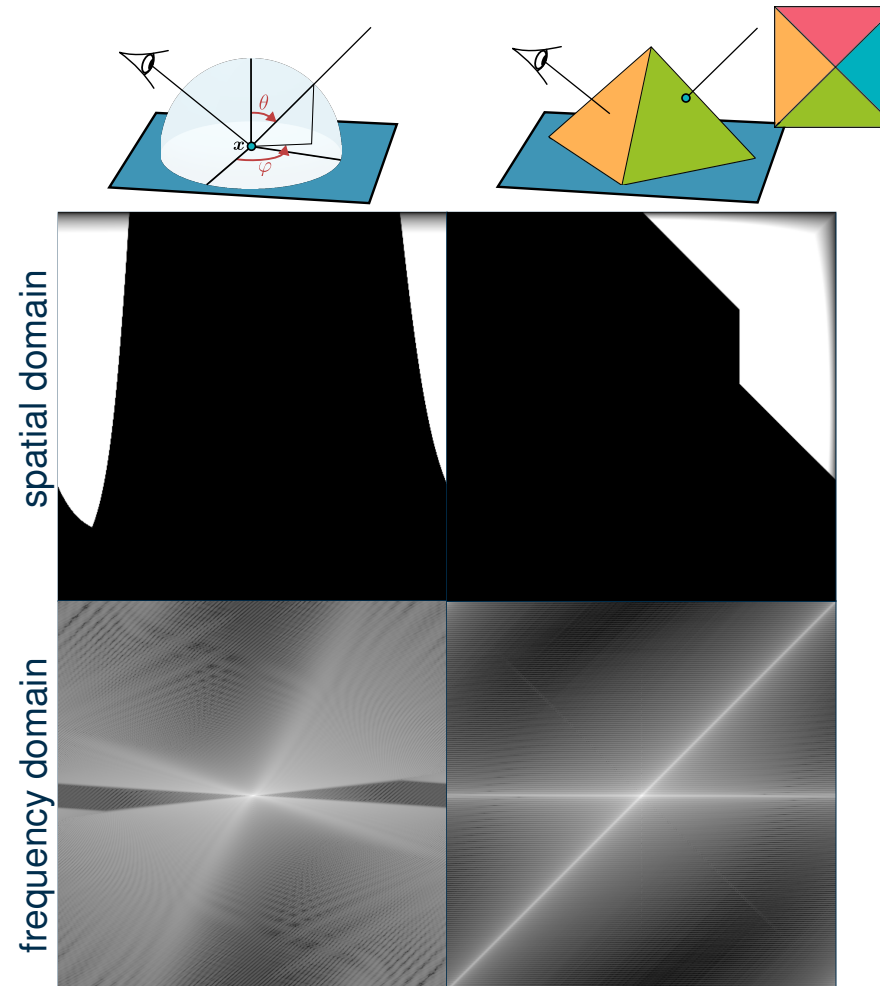
results for other scenes in the paper

Hemisphere Sampling



results (convergence plots) in the paper
Octahedron Environment Maps (Engelhardt et al., 2008)

Hemisphere Sampling

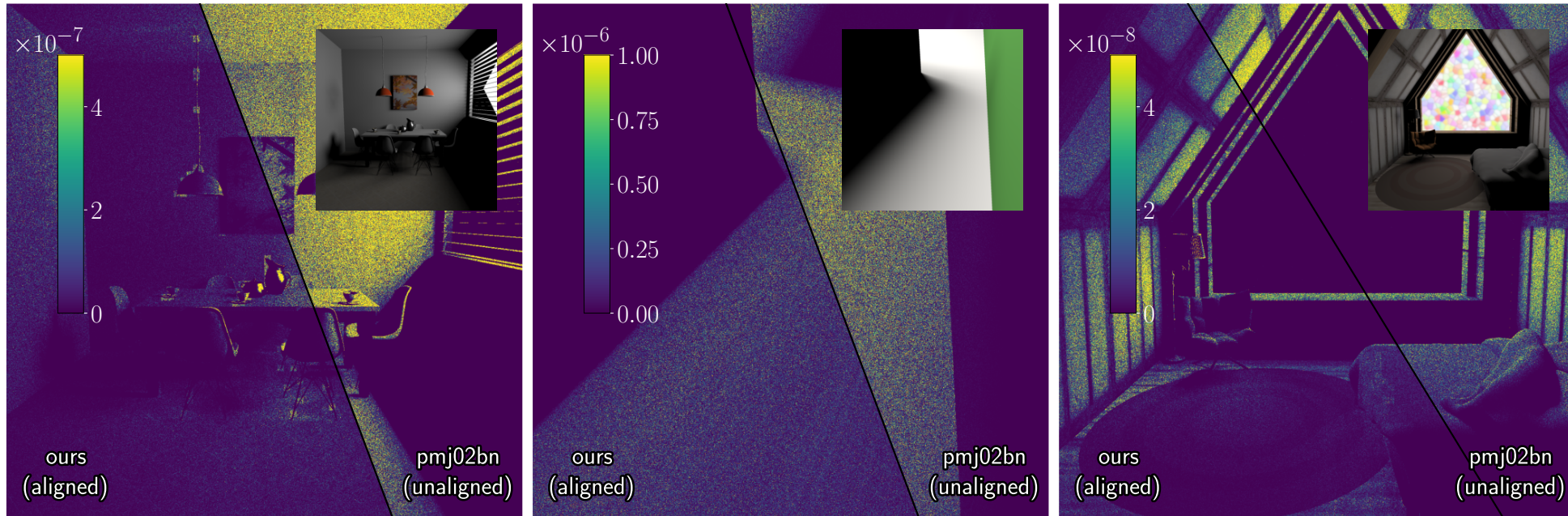


results (convergence plots) in the paper
Octahedron Environment Maps (Engelhardt et al., 2008)

Conclusion and Future Work

conclusion

- optimization of anisotropic sampling patterns (with >2 low-energy directions)
 - estimation of the orientation of discontinuities
- ⇒ better convergence rate on 2D discontinuous integrands (lower variance/error)



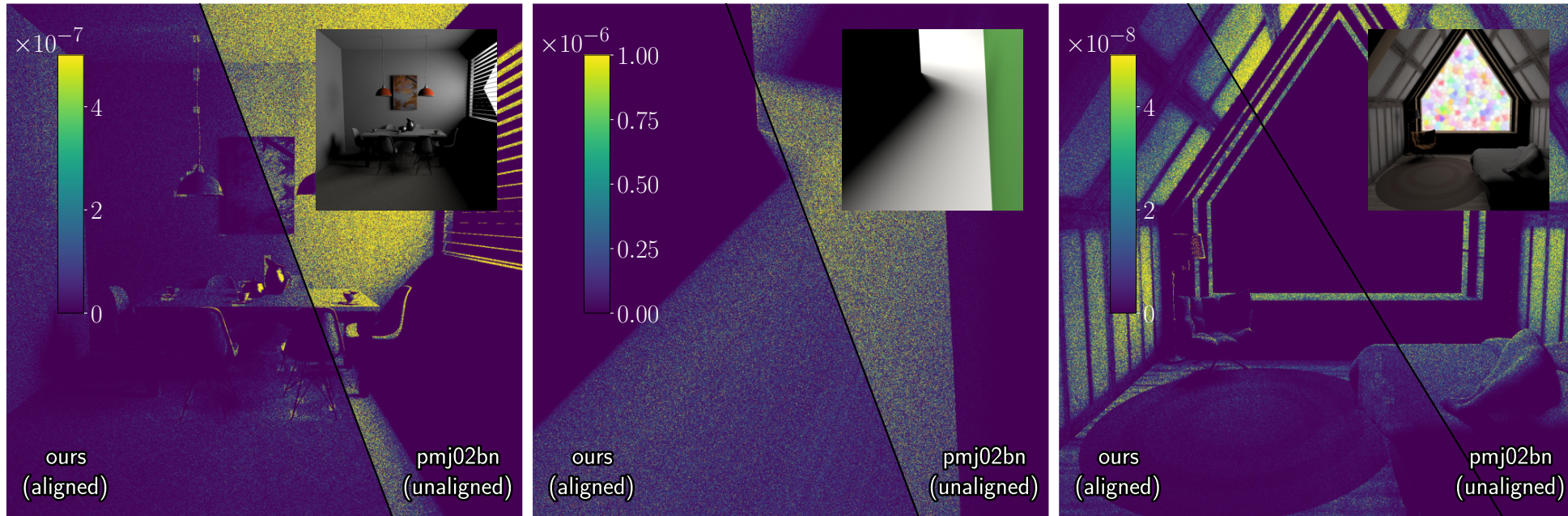
Conclusion and Future Work

conclusion

- optimization of anisotropic sampling patterns (with >2 low-energy directions)
 - estimation of the orientation of discontinuities
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future work

- online point optimization
- reduce overhead (16-64 additional spp) for principal direction estimation
- higher-dimensional transport



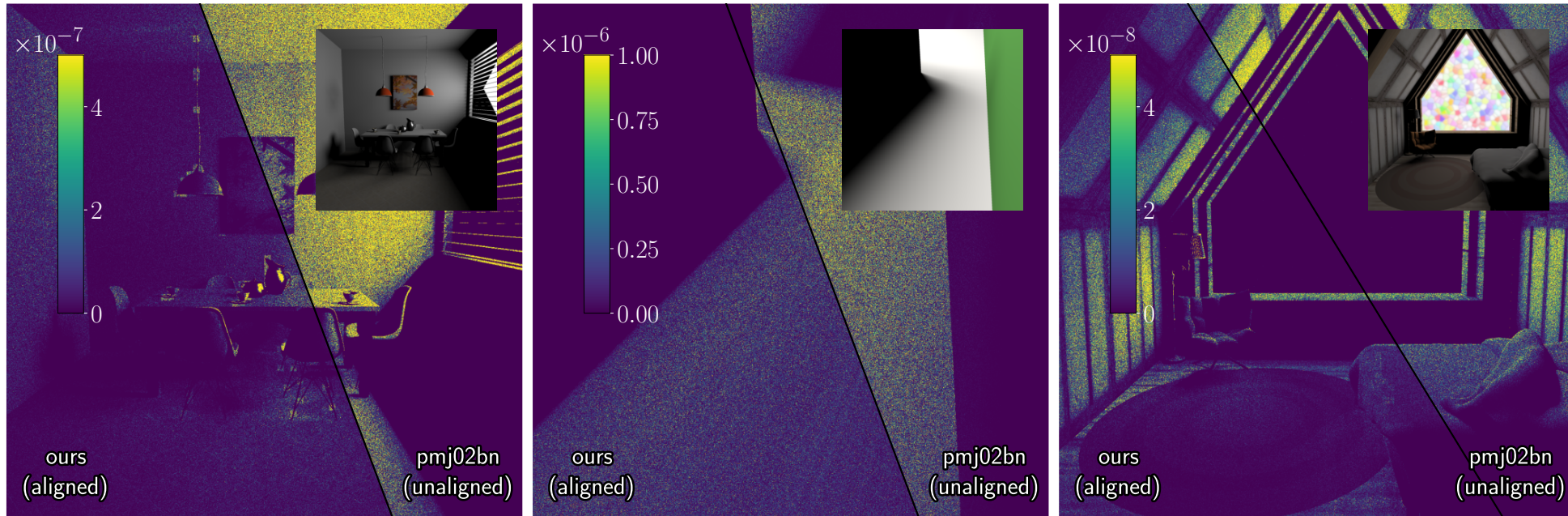
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Thank you!

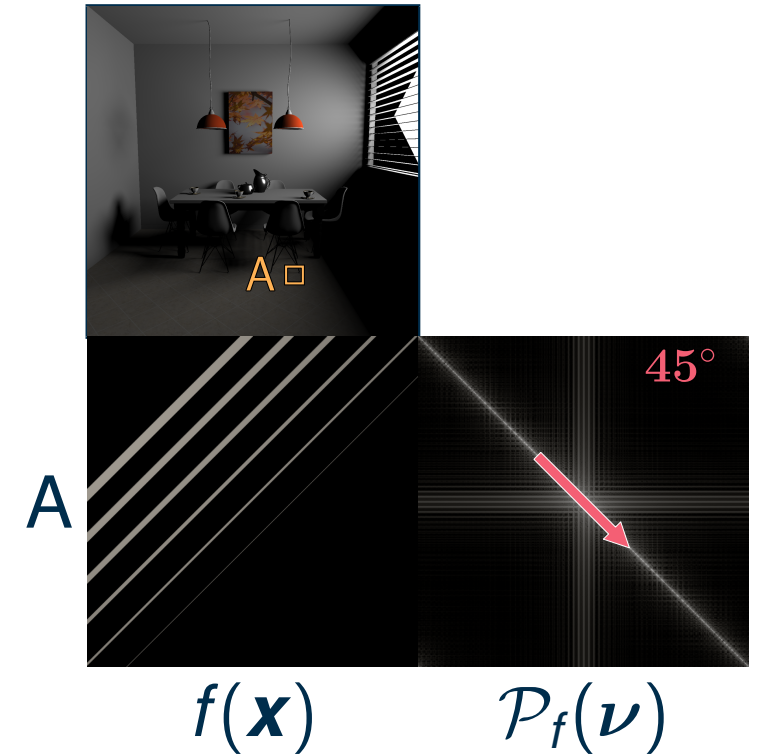
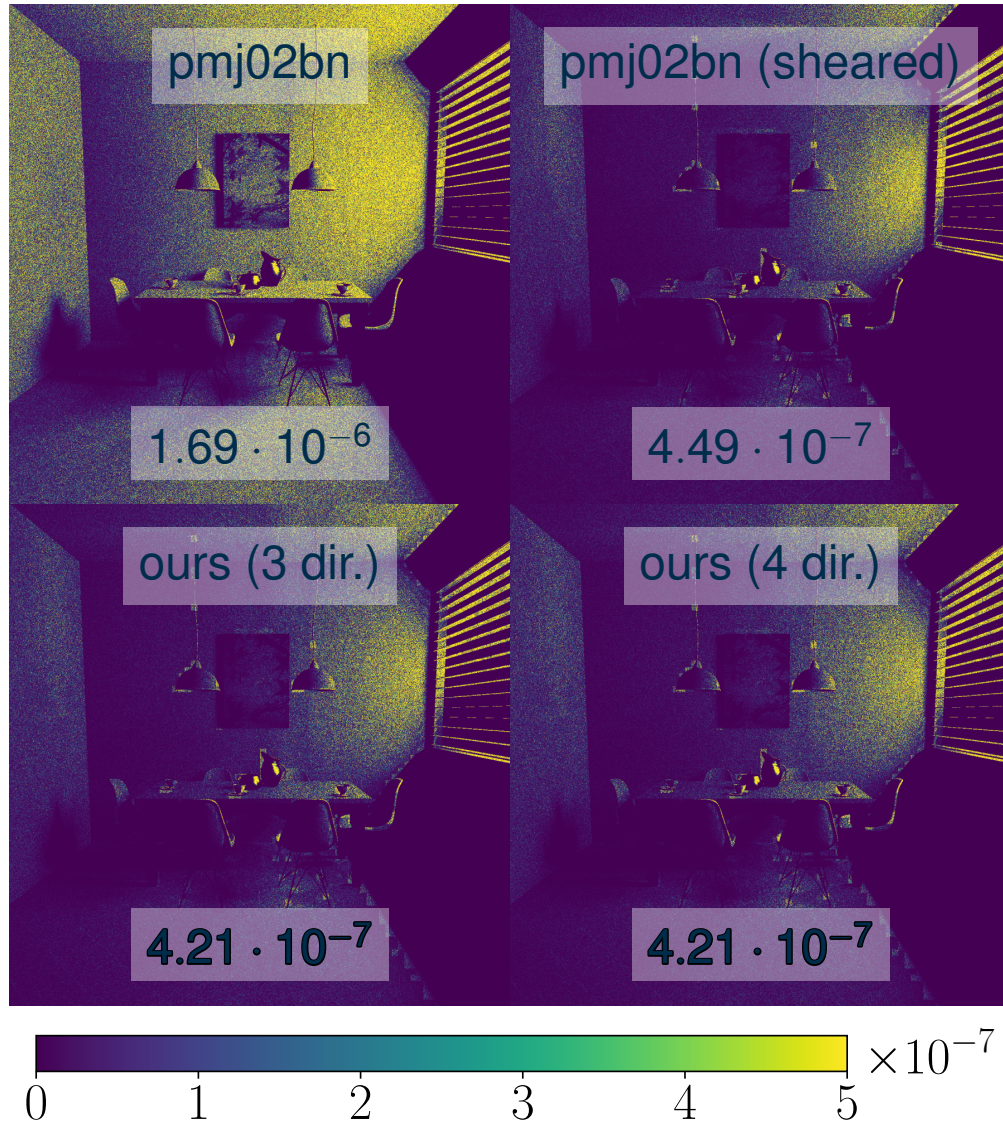
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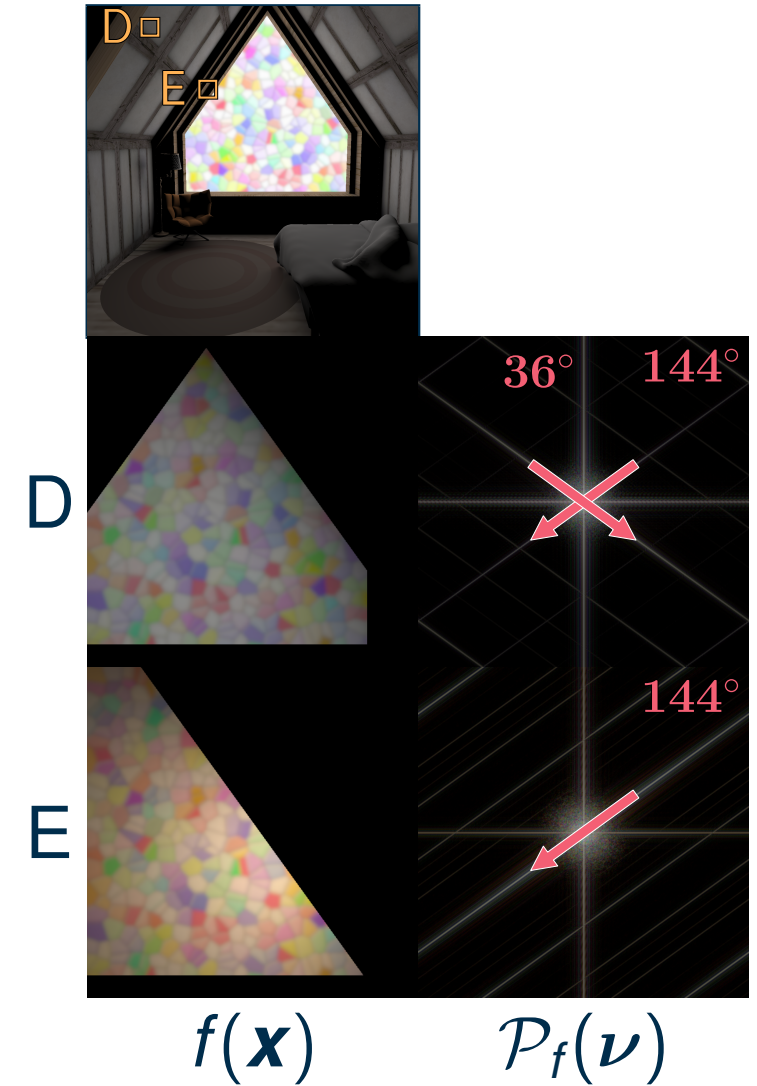
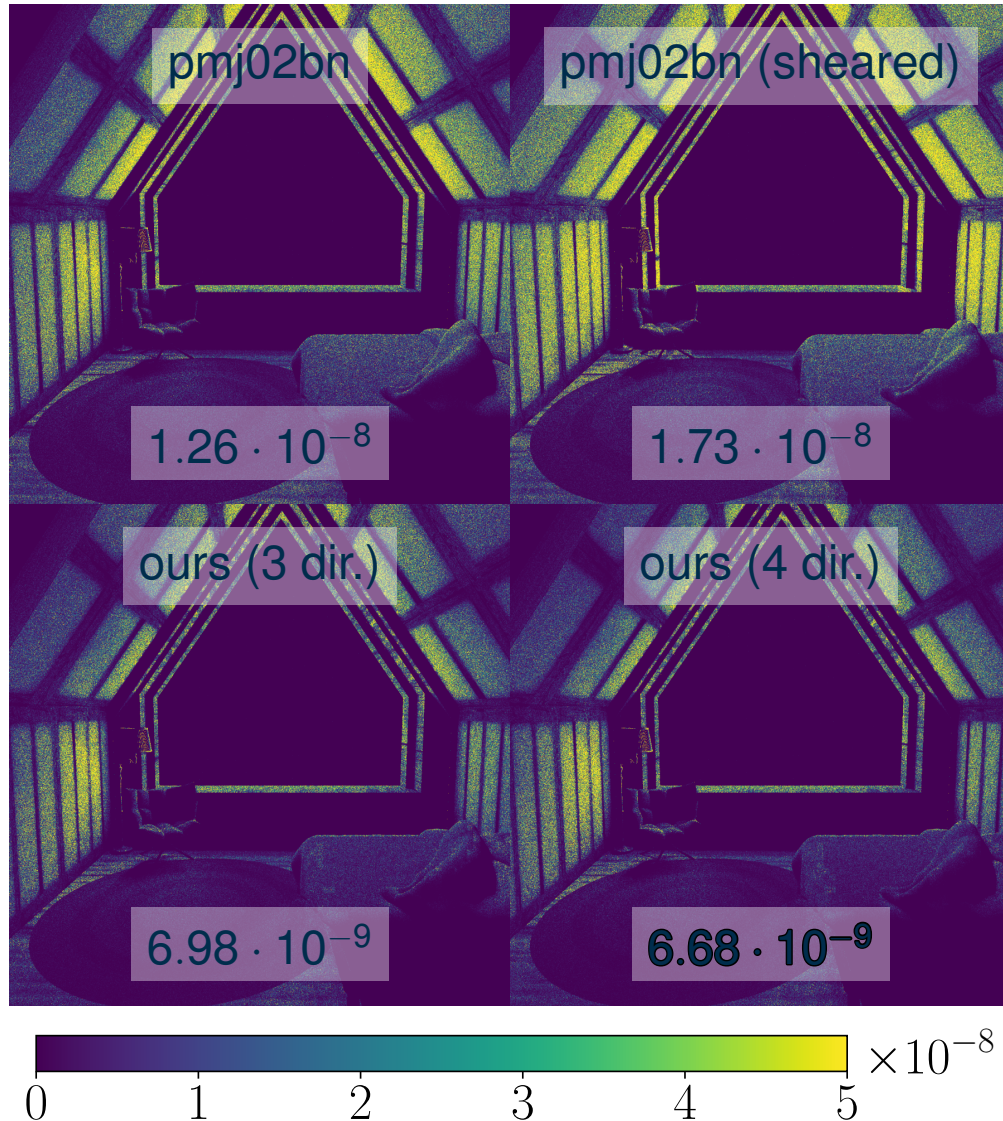
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Results: $\text{Var}(I_N)$, $N=4096$ spp, Area Light Sampling



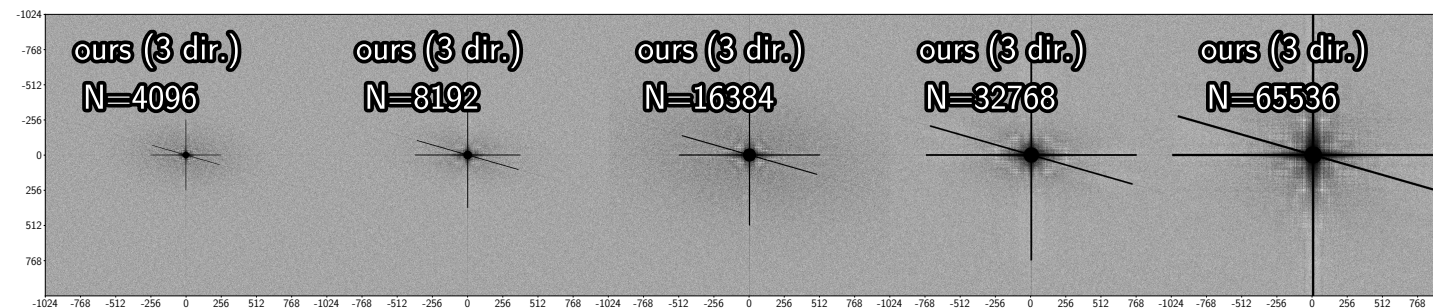
Results: $\text{Var}(I_N)$, $N=4096$ spp, Area Light Sampling



Point Optimization Parameters and Timings

PyTorch + CUDA on NVIDIA RTX 5080

#Points	#Frequencies (length, width, blue noise)	Optimization Time [min:s]
$2^{12} = 4096$	4303 (256, 5, 24)	00:17
$2^{13} = 8192$	9169 (384, 7, 32)	00:52
$2^{14} = 16384$	16099 (512, 9, 48)	02:49
$2^{15} = 32768$	32708 (768, 13, 56)	10:45
$2^{16} = 65536$	55741 (1024, 17, 64)	35:32



Loss Terms: Mathematical Details

- spectrum loss

- $\Lambda \subset \mathbb{Z}^2$ user-specified target

$$\mathcal{L}_{\text{spec}}(\mathcal{S}) = \frac{1}{|\Lambda|} \sum_{\boldsymbol{\nu} \in \Lambda} \|\mathcal{P}_{\mathcal{S}}(\boldsymbol{\nu}) - \mathcal{P}_{\text{TARGET}}(\boldsymbol{\nu})\|^2$$

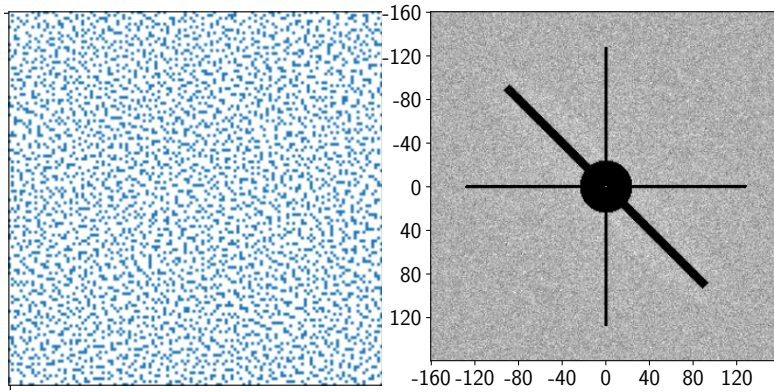
- sliced Wasserstein loss

- $\mathcal{A}$ contains the low-energy directions (e.g., 0° , 90° , 16°)

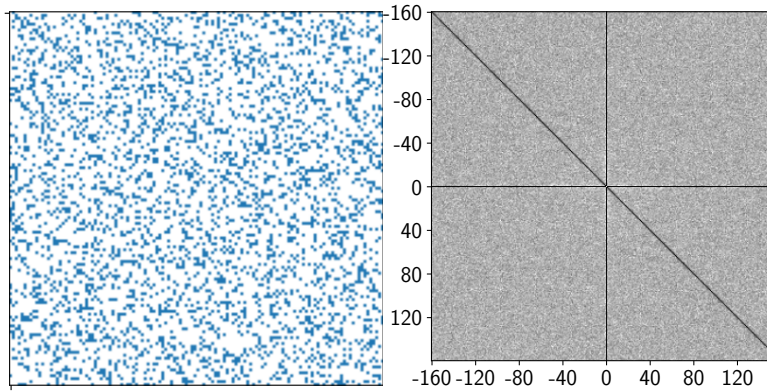
$$\mathcal{L}_{\text{SW1D}}(\mathcal{S}, \mathbf{d}) = \frac{1}{N} \|\text{sort}(\mathcal{S} \cdot \mathbf{d}) - D_{\text{TARGET}}(\mathbf{d})\|^2$$

$$\mathcal{L}_{\text{SW}}(\mathcal{S}) = \frac{1}{|\mathcal{A}|} \sum_{\mathbf{d} \in \mathcal{A}} \mathcal{L}_{\text{SW1D}}(\mathcal{S}, \mathbf{d})$$

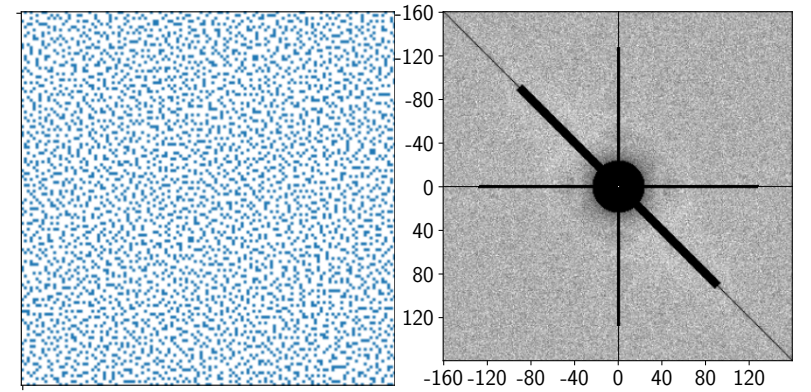
Loss Terms: Visual Comparison



(a) $\mathcal{L}_{\text{spec}}(S)$



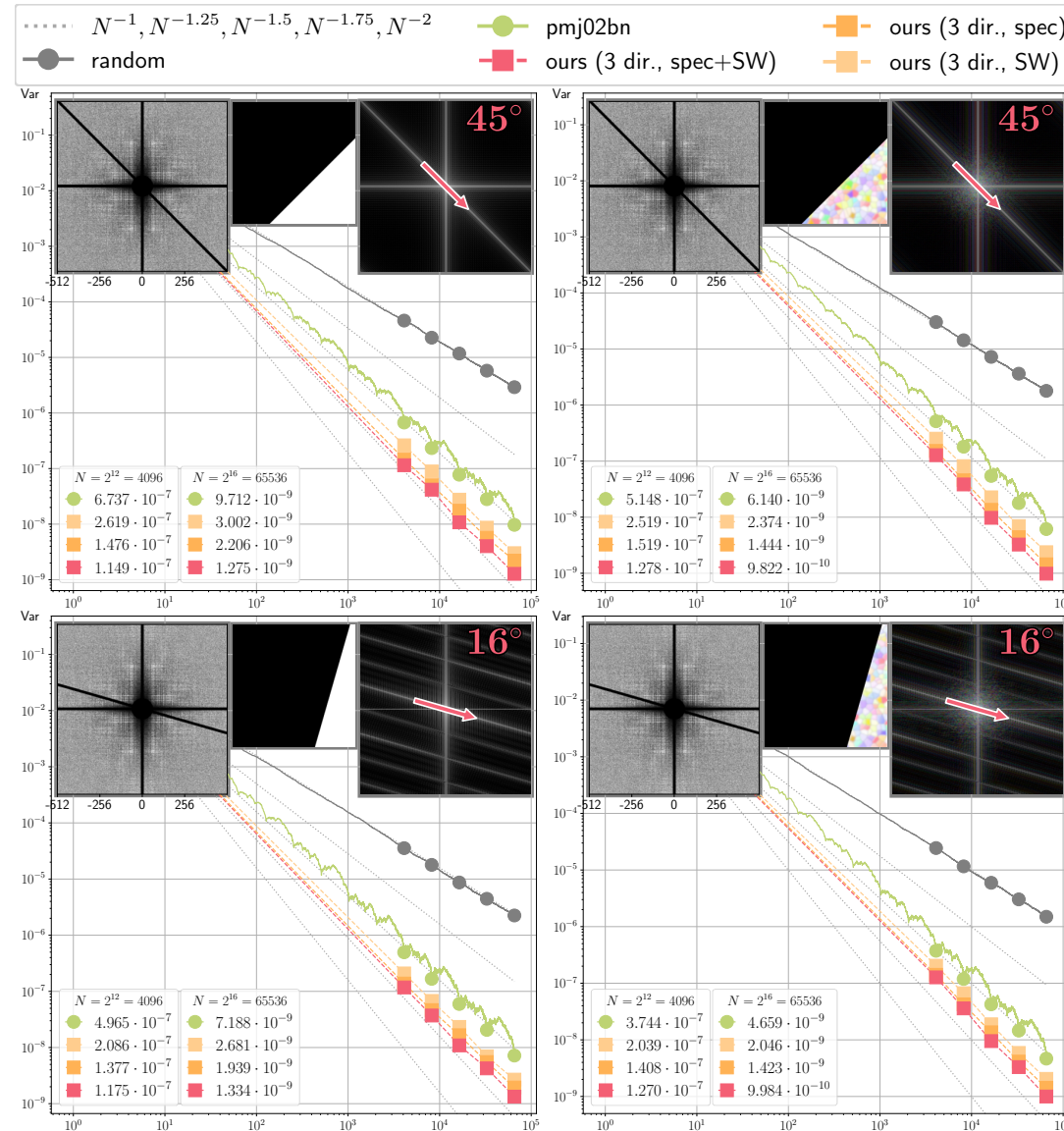
(b) $\mathcal{L}_{\text{SW}}(S)$



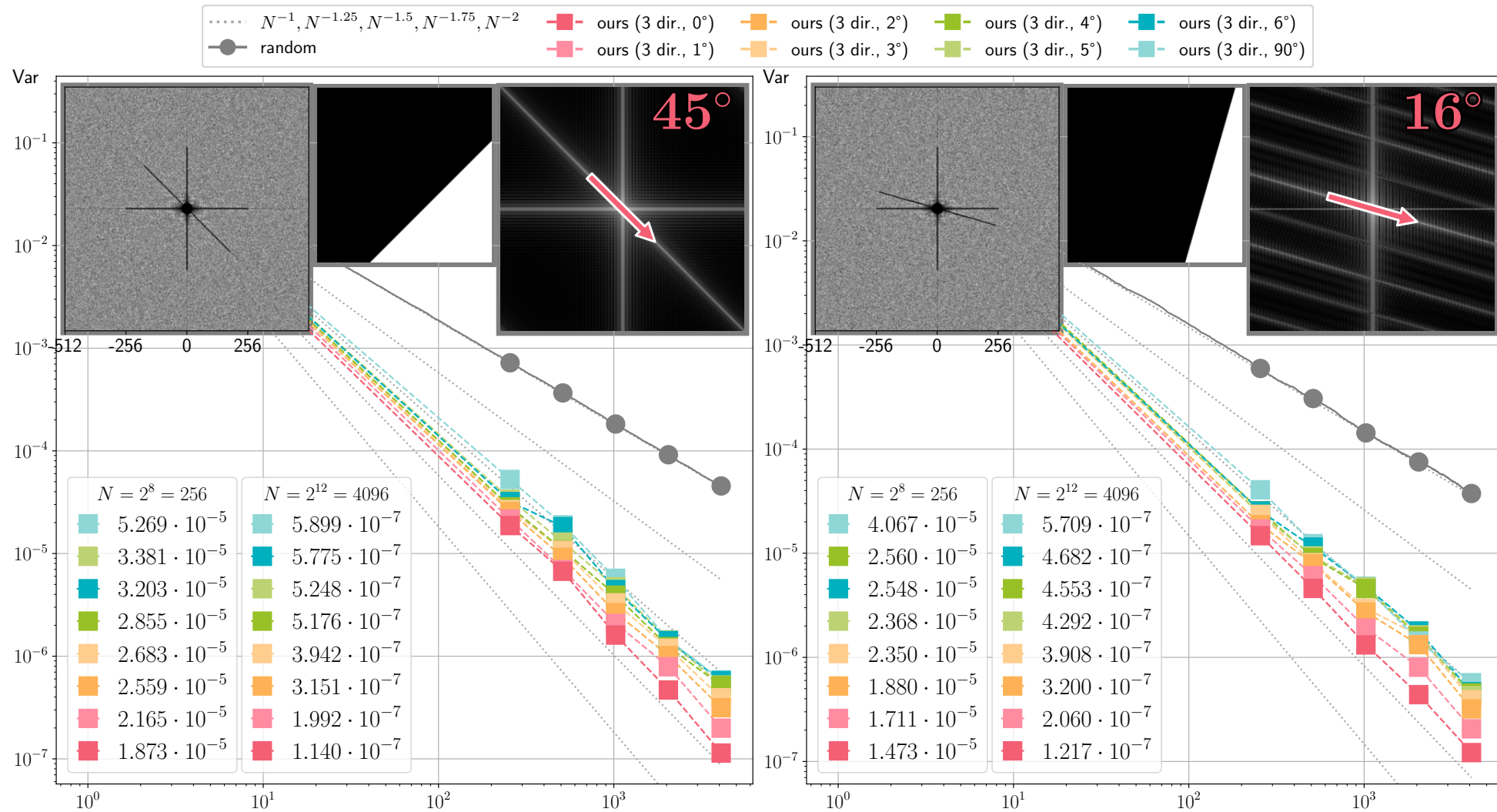
(c) $\mathcal{L}_{\text{spec}}(S) + c\mathcal{L}_{\text{SW}}(S)$

- While $\mathcal{L}_{\text{SW}}$ alone ensures correct marginal distributions of the points, it allows for clusters and holes to form in 2D (b).
- $\mathcal{L}_{\text{spec}}$ alone ensures evenly, blue-noise-like distributed points in 2D and suppresses a set of selected frequencies very effectively, but it cannot enforce directional stratification cheaply, and therefore fails to suppress increasingly higher frequencies (a).

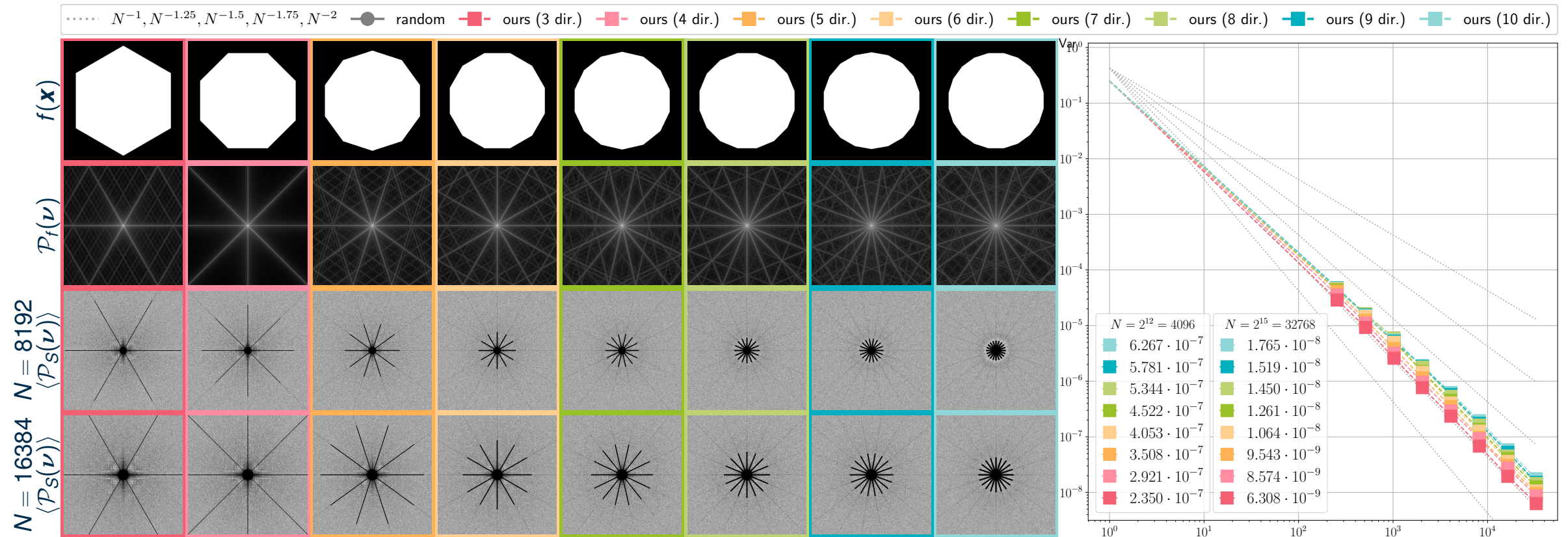
Loss Terms: Quantitative Comparison



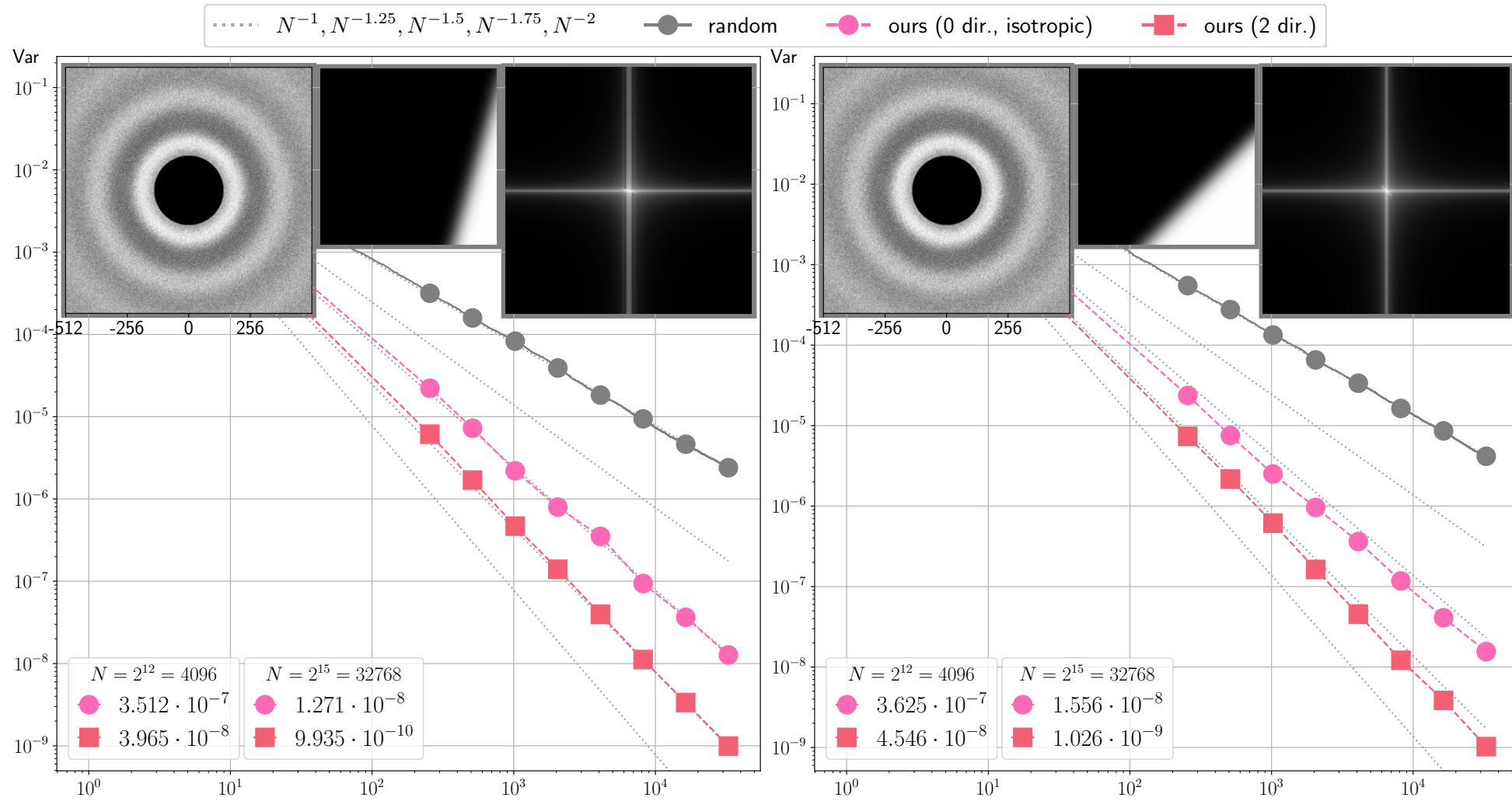
Misalignment



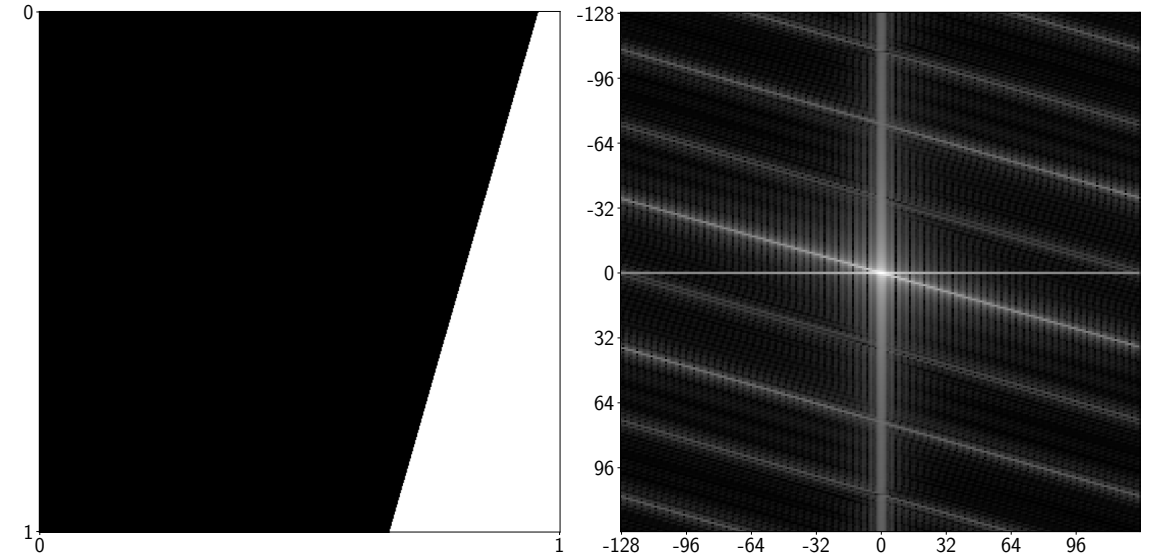
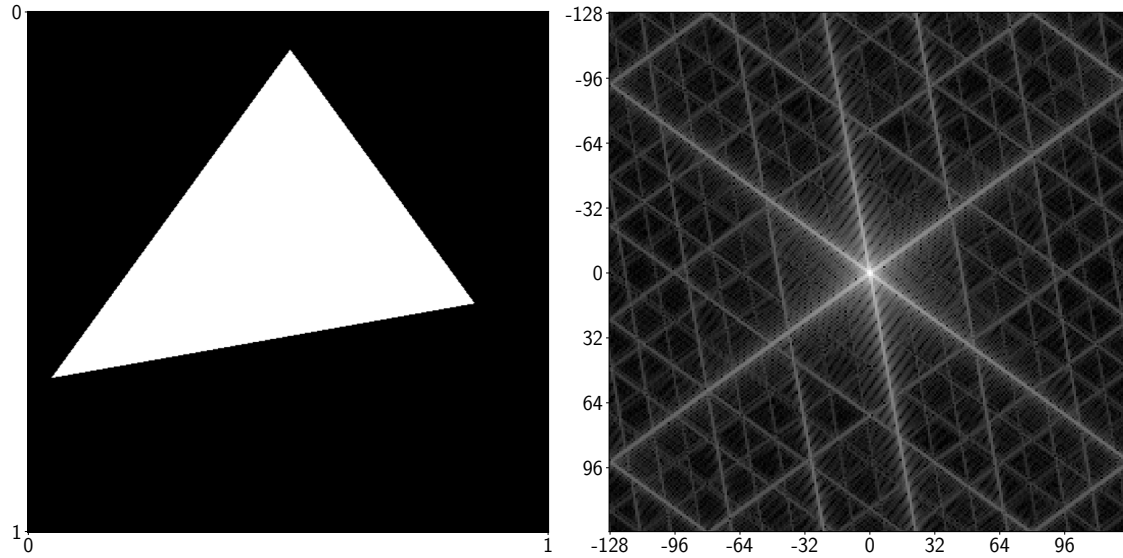
Increasing Number of Discontinuities



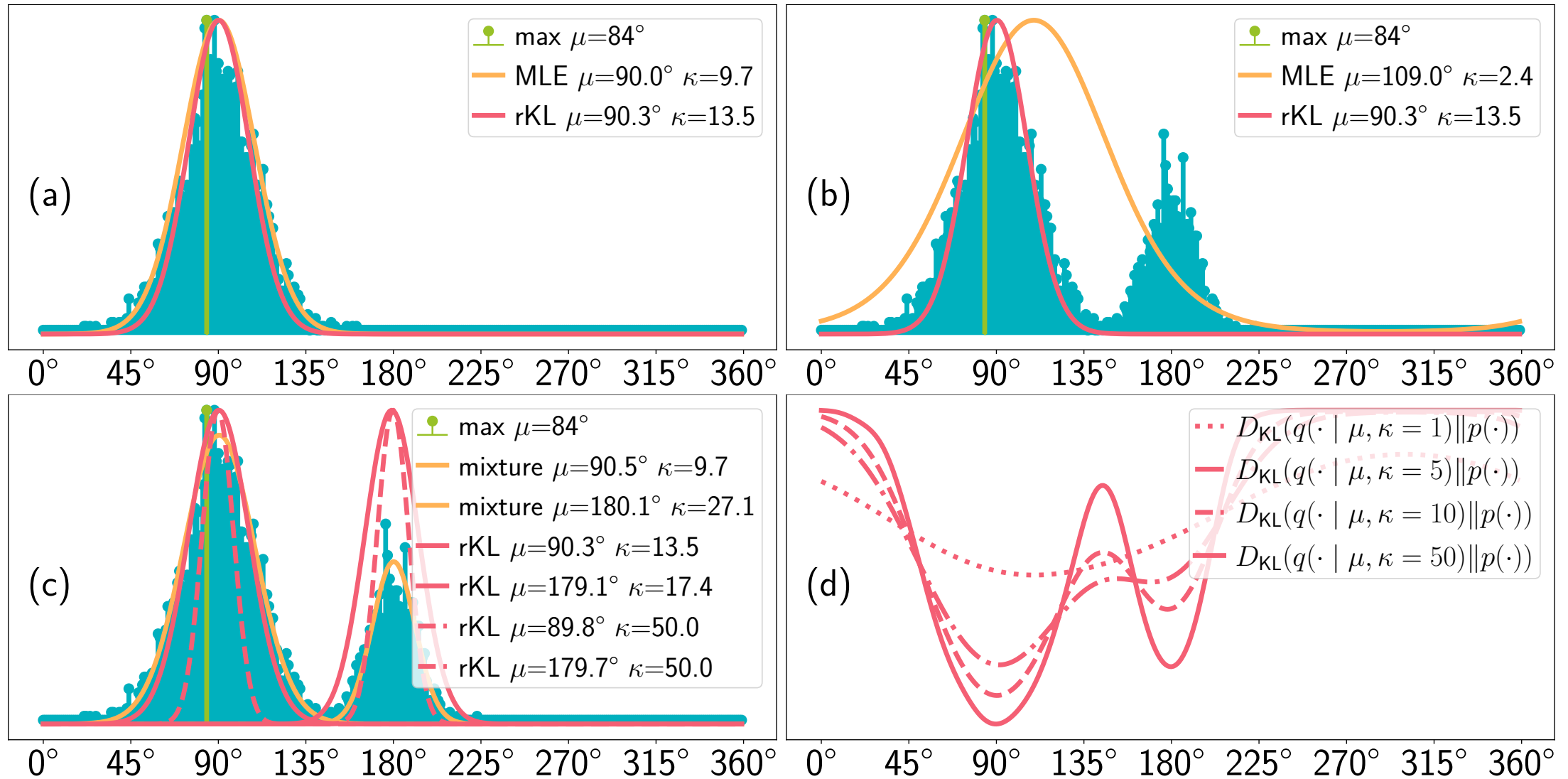
Isotropic vs. Anisotropic



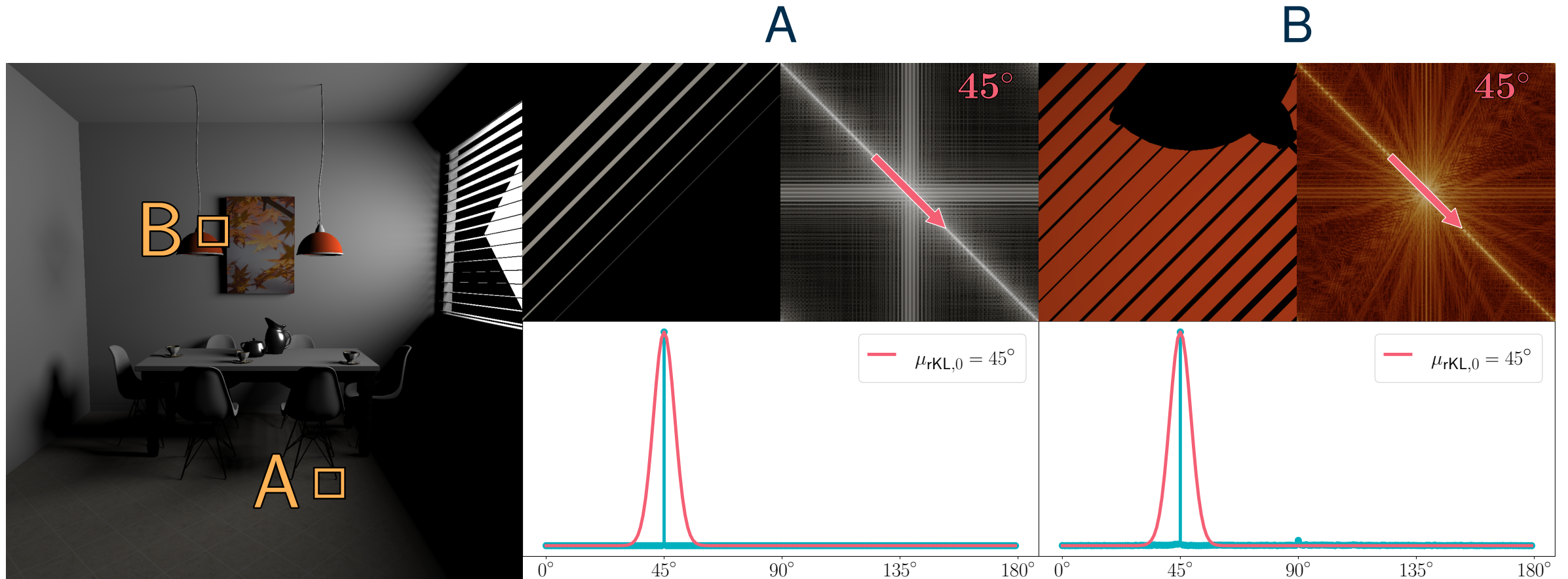
Integrands Without Horizontal and Vertical Energies



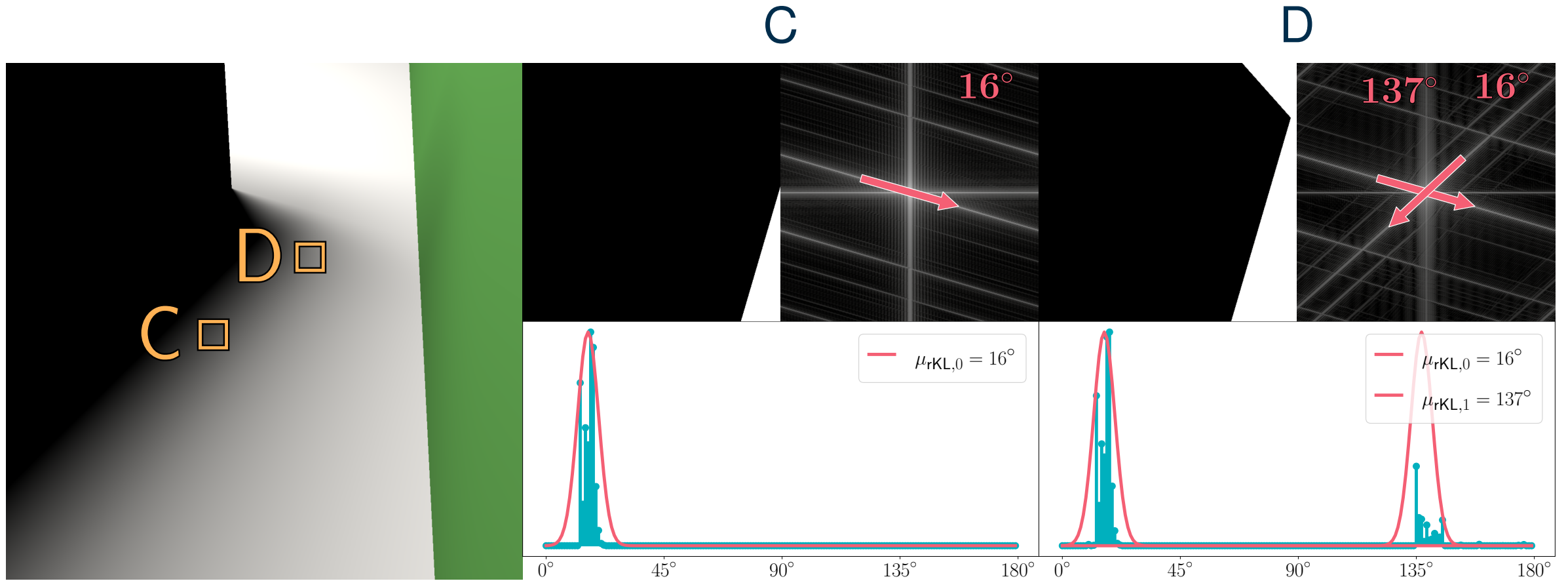
Principal Direction Estimation



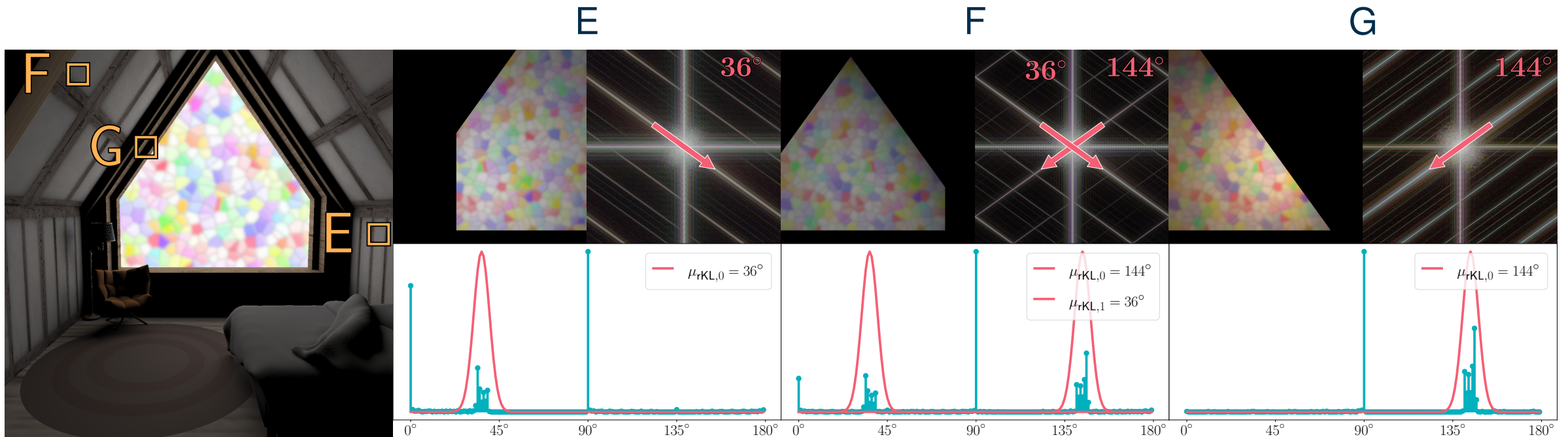
Results: Principal Direction Estimation



Results: Principal Direction Estimation



Results: Principal Direction Estimation



Smoothed Discontinuities

