Supplementary Document on Sampling Projected Spherical Caps in Real Time

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In this supplementary document, we provide derivations to underpin the individual steps of our algorithm. Furthermore, we prove that the variation of the sampled density is bounded.

1 GEOMETRIC QUANTITIES

The algorithms discussed in the paper compute various lengths and coordinates. In the following paragraphs, we provide derivations for the quantities computed in non-trivial ways.

Extent of the circle $\partial\mathbb{D}$ along the y-axis r_y. By construction, the y-axis is parallel to the plane of the circle and thus the extent r_y agrees with the radius of the circle. From Figure 1, the following relation is apparent:

$$r_y = \frac{r}{\|\boldsymbol{d}\|_2}$$

Extent of the circle $\partial\mathbb{D}$ along the x-axis r_x. The direction towards the sphere center $\boldsymbol{\omega}_d \in \mathbb{S}^2$ is the unit normal vector of the plane of the circle $\partial\mathbb{D}$. The cosine of its angle with the z-axis is $\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d$. This cosine is proportional to the area of the projection of the circle to the xy-plane, which is in turn proportional to its extent along the x-axis. The sign of the cosine becomes negative if and only if the center is below the horizon. For a cosine of one, the projection is still a circle and $r_x = r_y$. Therefore,

$$r_x = (\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d)r_y.$$

Extent of the circle $\partial\mathbb{D}$ along the z-axis r_z. By the same argument as for r_x, we have

$$r_z = (\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)r_y.$$

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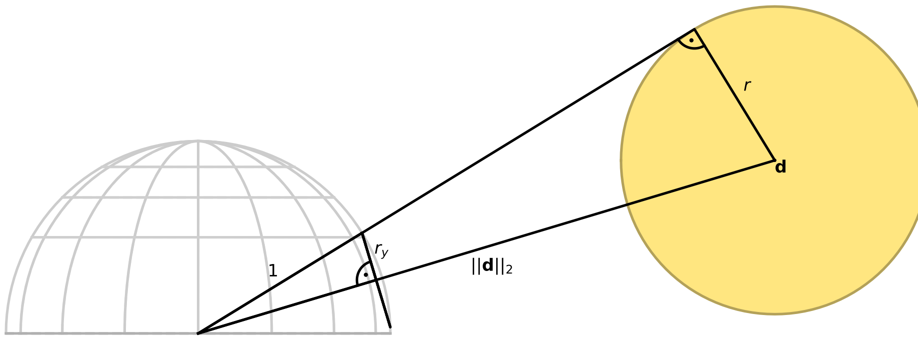

Fig. 1. The radius of the sphere r and the distance to the sphere center $\|\boldsymbol{d}\|_2$ are opposite and hypothenuse of a right triangle. The radius r_y of the circle $\partial\mathbb{D}$ is the opposite of a right triangle with hypothenuse one and the same opening angle.

Center $\boldsymbol{c} = (c_x, 0, c_z)$ *of the circle* $\partial\mathbb{D}$. The local coordinates $(c_x, r_y, c_z)^\mathsf{T}$ must describe a point on the circle $\partial\mathbb{D}$, which is a subset of the unit sphere. Thus

$$c_x^2 + r_y^2 + c_z^2 = 1 \qquad \Rightarrow \qquad c_x^2 + c_z^2 = 1 - r_y^2. \tag{1}$$

Since $\boldsymbol{c}$ and $\boldsymbol{\omega}_d$ describe two points on a common line through the origin (but in different coordinate systems), we know

$$\frac{c_z}{c_x} = \frac{\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d}{\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d} \qquad \Rightarrow \qquad c_z = \frac{\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d}{\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d} c_x.$$

Substituting this result into Equation (1) we find:

$$\begin{aligned} & c_x^2 + c_z^2 = 1 - r_y^2 \\ \Rightarrow & \left(1 + \frac{(\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d)^2}{(\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)^2}\right) c_x^2 = 1 - r_y^2 \\ \Rightarrow & \frac{(\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)^2 + (\boldsymbol{y}^\mathsf{T}\boldsymbol{\omega}_d)^2 + (\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d)^2}{(\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)^2} c_x^2 = 1 - r_y^2 \\ \Rightarrow & c_x^2 = (\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)^2(1 - r_y^2) \\ \Rightarrow & c_x = (\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)\sqrt{1 - r_y^2} \end{aligned}$$

The derivation for c_z is analogue. Note that the sign of c_z agrees with the sign of $\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d$.

The tangent point $\boldsymbol{t} = (t_x, t_y, 0)^\mathsf{T}$. The tangent point lies on the horizon, so $t_x^2 + t_y^2 = 1$. It also lies on the circle $\partial\mathbb{D}$, i.e.

$$r_y^2 = (t_x - c_x)^2 + t_y^2 + c_z^2 = (t_x - c_x)^2 + 1 - t_x^2 + c_z^2 = -2t_x c_x + c_x^2 + 1 + c_z^2. \tag{2}$$

The normalization factor u turns out to be closely related to $\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d$:

$$\frac{1}{u^2} = \|\boldsymbol{\omega}_d - \boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d\boldsymbol{z}\|_2^2 = 1 - 2\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d\boldsymbol{\omega}_d^\mathsf{T}\boldsymbol{z} + (\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d)^2\|\boldsymbol{z}\|_2^2 = 1 - (\boldsymbol{z}^\mathsf{T}\boldsymbol{\omega}_d)^2 = (\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d)^2$$

We could have computed $u = \frac{1}{\boldsymbol{x}^\mathsf{T}\boldsymbol{\omega}_d}$ on the spot but reusing the readily available normalization factor is more efficient. Now if we substitute the formula for t_x from the Algorithm into Equation (2), we find that it solves it:

$$-2uvc_x + c_x^2 + 1 + c_z^2 = -2(1 - r_y^2) + c_x^2 + 1 + c_z^2 = 2r_y^2 - (1 - c_x^2 - c_z^2) = 2r_y^2 - r_y^2 = r_y^2$$

The sign of t_x is always positive.

The scaling along the y-axis s_y. The extent of the semicircle along the y-axis at height d_z is $2\sqrt{t_y^2 - d_z^2}$. This is also the extent for the scaled semicircle at height $\omega_{i,z}$. The ellipse in the yz-plane has the the implicit representation

$$\frac{\omega_{i,y}^2}{r_y^2} + \frac{(\omega_{i,z} - c_z)^2}{r_z^2} \leq 1.$$

Solving for $\omega_{i,y}$, we obtain the extent of the ellipse along the y-axis at height $\omega_{i,z}$:

$$2r_y\sqrt{1 - \frac{(\omega_{i,z} - c_z)^2}{r_z^2}}$$

The scaling has to be the quotient of these two widths

$$s_y = \frac{2r_y\sqrt{1 - \frac{(\omega_{i,z}-c_z)^2}{r_z^2}}}{2\sqrt{t_y^2 - d_z^2}} = r_y\sqrt{\frac{1 - \frac{(\omega_{i,z}-c_z)^2}{r_z^2}}{t_y^2 - d_z^2}} = r_y\sqrt{\frac{r_z^2 - (\omega_{i,z} - c_z)^2}{r_z^2(t_y^2 - d_z^2)}}.$$

2 COMPUTING THE DENSITY

In Cases 1 and 2, the projected spherical cap is sampled uniformly and thus the density is simply the reciprocal of the projected solid angle. In Case 3, the density depends on the random numbers. To compute it, we have to consider the determinant of the Jacobian of the warp.

We begin by defining this warp formally. The first step is to project points from the xy-plane onto the upper hemisphere and into the yz-plane:

$$\varphi(x, y) := (y, \sqrt{1 - x^2 - y^2})^\mathsf{T}$$

The second step is to scale uniformly along the z-axis and non-uniformly along the y-axis. The scaling along the y-axis is given by

$$s_y(z) := r_y\sqrt{\frac{r_z^2 - \left(\frac{c_z+r_z}{t_y}z - c_z\right)^2}{r_z^2(t_y^2 - z^2)}}.$$

The overall scaling is

$$\psi(y, z) := \left(s_y(z)y, \frac{c_z + r_z}{t_y}z\right)^\mathsf{T}.$$

Finally, φ^{-1} maps points back to the xy-plane. The overall warping transform is $\theta := \varphi^{-1} \circ \psi \circ \varphi$.

To compute the Jacobian determinant of the entire warp, we first compute the determinant for each step.

$$\det J_\varphi(x,y) = \det \begin{pmatrix} \frac{\partial}{\partial x} y & \frac{\partial}{\partial y} y \\ \frac{\partial}{\partial x}\sqrt{1-x^2-y^2} & \frac{\partial}{\partial y}\sqrt{1-x^2-y^2} \end{pmatrix}$$

$$= -\frac{\partial}{\partial x}\sqrt{1-x^2-y^2} = 2x\frac{1}{2}\frac{1}{\sqrt{1-x^2-y^2}} = \frac{x}{\sqrt{1-x^2-y^2}}$$

$$\det J_\psi(y,z) = \det \begin{pmatrix} \frac{\partial}{\partial y} s_y(z)y & \frac{\partial}{\partial z} s_y(z)y \\ \frac{\partial}{\partial y}\frac{c_z+r_z}{t_y}z & \frac{\partial}{\partial z}\frac{c_z+r_z}{t_y}z \end{pmatrix} = \det \begin{pmatrix} s_y(z) & \frac{\partial}{\partial z} s_y(z)y \\ 0 & \frac{c_z+r_z}{t_y} \end{pmatrix} = \frac{c_z+r_z}{t_y} s_y(z)$$

$$\det J_{\varphi^{-1}}(y,z) = \frac{1}{\det J_\varphi(\varphi^{-1}(y,z))}$$

In accordance with the algorithm, we adopt the notions

$$d_x, d_y \in \mathbb{R}, \quad d_z := \sqrt{1-d_x^2-d_y^2}, \quad (\omega_{i,x}, \omega_{i,y})^\mathsf{T} := \theta(d_x, d_y), \quad \omega_{i,z} := \sqrt{1-\omega_{i,x}^2-\omega_{i,y}^2}.$$

This allows us to rewrite the determinant of the Jacobian of the warp:

$$\det J_\theta(d_x,d_y) = \frac{1}{\det J_\varphi(\omega_{i,x},\omega_{i,y})} \det J_\psi(d_y,d_z) \det J_\varphi(d_x,d_y)$$

$$= \frac{\omega_{i,z}}{\omega_{i,x}} \frac{c_z+r_z}{t_y} s_y(d_z) \frac{d_x}{d_z} = \frac{c_z+r_z}{t_y} \frac{\omega_{i,z}}{d_z} \frac{d_x s_y(d_z)}{\omega_{i,x}}$$

$$= \frac{(c_z+r_z)^2}{t_y^2} \frac{d_x s_y(d_z)}{\omega_{i,x}}$$

To obtain the final density, we have to divide the density in the unwarped cut disk by the absolute value of the Jacobian determinant. The Jacobian determinant is always negative because $d_x < 0$. Thus, we arrive at the formula used in the algorithm:

$$p(\xi_0,\xi_1) = \frac{1}{A_D} \frac{1}{-\det J_\theta(d_x,d_y)} = -\frac{t_y^2}{(c_z+r_z)^2 A_D} \frac{\omega_{i,x}}{d_x s_y(d_z)}$$

3 BOUNDING THE VARIATION OF THE DENSITY

In the paper we claim that for a fixed spherical cap in Case 3, the density obeys the following bound:

$$\frac{\max_{\xi_0,\xi_1\in[0,1]} p(\xi_0,\xi_1)}{\min_{\xi_0,\xi_1\in[0,1]} p(\xi_0,\xi_1)} \leq \frac{c_x - r_x}{t_x}\sqrt{\frac{t_x-(c_x+r_x)}{-r_x}}$$

To prove this statement, we first observe that the random numbers parameterize the cut disk

$$\mathbb{K} := \{(x,y)^\mathsf{T} \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \ \wedge \ x \leq -t_x\}.$$

Furthermore, we can drop all factors that only depend on the spherical cap because they cancel out in the quotient. Therefore, what we are actually interested in is the function

$$\frac{\theta_x(x,y)}{-x s_y(\sqrt{1-x^2-y^2})}$$

for $x, y \in \mathbb{K}$ where $\theta_x(x,y)$ denotes the x-component of $\theta(x,y) \in \mathbb{R}^2$.

3.1 Bounding the Scaling Along the y-axis s_y

As a first step, we compute bounds on $s_y(z)$ for $z \in [0, t_y)$. It helps to simplify the expression for $s_y(z)$ as follows:

$$\begin{aligned}\frac{t_y^2 r_z^2}{r_y^2} s_y^2(z) &= t_y^2 r_z^2 \frac{r_z^2 - \left(\frac{c_z+r_z}{t_y} z - c_z\right)^2}{r_z^2(t_y^2 - z^2)} = \frac{t_y^2 r_z^2 - ((c_z + r_z)z - t_y c_z)^2}{t_y^2 - z^2} \\ &= \frac{t_y^2 r_z^2 - (c_z + r_z)^2 z^2 + 2(c_z + r_z) t_y c_z z - t_y^2 c_z^2}{t_y^2 - z^2} \\ &= (c_z + r_z)\frac{t_y^2(r_z - c_z) - (c_z + r_z) z^2 + 2 t_y c_z z}{t_y^2 - z^2}\end{aligned}$$

Now we prove lower and upper bounds:

$$\begin{aligned}\frac{t_y^2 r_z^2}{r_y^2} s_y^2(z) - (c_z + r_z) r_z &= (c_z + r_z)\frac{t_y^2(r_z - c_z) - (c_z + r_z) z^2 + 2 t_y c_z z - r_z(t_y^2 - z^2)}{t_y^2 - z^2} \\ &= (c_z + r_z)\frac{-t_y^2 c_z - c_z z^2 + 2 t_y c_z z}{t_y^2 - z^2} = -c_z(c_z + r_z)\frac{t_y^2 - 2 t_y z + z^2}{t_y^2 - z^2} \\ &= -c_z(c_z + r_z)\frac{(t_y - z)^2}{t_y^2 - z^2} \geq 0 \end{aligned} \tag{3}$$

$$\begin{aligned}\frac{t_y^2 r_z^2}{r_y^2} s_y^2(z) - (r_z^2 - c_z^2) &= (c_z + r_z)\frac{t_y^2(r_z - c_z) - (c_z + r_z) z^2 + 2 t_y c_z z - (r_z - c_z)(t_y^2 - z^2)}{t_y^2 - z^2} \\ &= (c_z + r_z)\frac{-2 c_z z^2 + 2 t_y c_z z}{t_y^2 - z^2} = -2 c_z (c_z + r_z) z \frac{z - t_y}{t_y^2 - z^2} \leq 0\end{aligned}$$

We further simplify the upper bound by utilizing the geometric relations from Section 1:

$$\begin{aligned}\frac{r_y^2}{t_y^2 r_z^2}(r_z^2 - c_z^2) &= \frac{1}{t_y^2 (\boldsymbol{x}^\top \boldsymbol{\omega}_d)^2}((\boldsymbol{x}^\top \boldsymbol{\omega}_d)^2 r_y^2 - (\boldsymbol{z}^\top \boldsymbol{\omega}_d)^2 (1 - r_y^2)) \\ &= \frac{r_y^2 - (\boldsymbol{z}^\top \boldsymbol{\omega}_d)^2}{(1 - t_x^2)(\boldsymbol{x}^\top \boldsymbol{\omega}_d)^2} = \frac{r_y^2 - (\boldsymbol{z}^\top \boldsymbol{\omega}_d)^2}{(\boldsymbol{x}^\top \boldsymbol{\omega}_d)^2 - (1 - r_y^2)} = 1\end{aligned}$$

This simplification also carries over to the lower bound:

$$\frac{r_y^2}{t_y^2 r_z^2}(c_z + r_z) r_z = \frac{r_y^2}{t_y^2 r_z^2}(r_z^2 - c_z^2)\frac{r_z}{r_z - c_z} = \frac{r_z}{r_z - c_z}$$

Hence, we have proven

$$\sqrt{\frac{r_z}{r_z - c_z}} \leq s_y(z) \leq 1.$$

3.2 Bounding the Quotient of x-Coordinates

Since $(t_x, t_y, 0)^\top$ and $(c_x - r_x, 0, c_z + r_z)^\top$ have unit length, we can bound the scaling along the z-axis:

$$\frac{(c_z + r_z)^2}{t_y^2} = \frac{1 - (c_x - r_x)^2}{1 - t_x^2} \leq \frac{1 - (c_x - r_x)^2}{1 - (c_x - r_x)^2} = 1$$

Combined with the bound $s_y \leq 1$, this leads to a lower bound for $\theta_x(x, y)$:

$$\begin{aligned}\theta_x^2(x,y) &= 1 - \|\psi(y, \sqrt{1-x^2-y^2})\|_2^2 \\ &= 1 - s_y^2(\sqrt{1-x^2-y^2})y^2 - \frac{(c_z+r_z)^2}{t_y^2}(1-x^2-y^2) \\ &\geq 1 - y^2 - (1-x^2-y^2) = x^2\end{aligned}$$

To also bound $\theta_x(x, y)$ from above, we reorganize terms:

$$\begin{aligned}\theta_x^2(x,y) &= 1 - s_y^2(\sqrt{1-x^2-y^2})y^2 - \frac{(c_z+r_z)^2}{t_y^2}(1-x^2-y^2) \\ &= 1 - \frac{(c_z+r_z)^2}{t_y^2} + \frac{(c_z+r_z)^2}{t_y^2}x^2 - \left(s_y^2(\sqrt{1-x^2-y^2}) - \frac{(c_z+r_z)^2}{t_y^2}\right)y^2\end{aligned}$$

Using the lower bound on s_y from Equation (3):

$$\begin{aligned}s_y^2(\sqrt{1-x^2-y^2}) - \frac{(c_z+r_z)^2}{t_y^2} &\geq \frac{r_y^2}{t_y^2 r_z^2}(c_z+r_z)r_z - \frac{(c_z+r_z)^2}{t_y^2} = (c_z+r_z)\frac{r_y^2 - (c_z+r_z)r_z}{t_y^2 r_z} \\ &\geq (c_z+r_z)\frac{r_y^2 - r_z^2}{t_y^2 r_z} \geq 0\end{aligned}$$

Hence, $\theta_x^2(x, y)$ is monotonically decreasing with y^2 and therefore:

$$\theta_x^2(x,y) \leq 1 - \frac{(c_z+r_z)^2}{t_y^2} + \frac{(c_z+r_z)^2}{t_y^2}x^2 = \frac{t_y^2 - (c_z+r_z)^2}{t_y^2} + \frac{(c_z+r_z)^2}{t_y^2}x^2$$

With that, we are ready to compute an upper bound for $\frac{\theta_x(x,y)}{-x}$:

$$\begin{aligned}\frac{\theta_x^2(x,y)}{x^2} &\leq \frac{t_y^2 - (c_z+r_z)^2}{x^2 t_y^2} + \frac{(c_z+r_z)^2}{t_y^2} \leq \frac{t_y^2 - (c_z+r_z)^2}{t_x^2 t_y^2} + \frac{(c_z+r_z)^2}{t_y^2} \\ &= \frac{t_y^2 - (1-t_x^2)(c_z+r_z)^2}{t_x^2 t_y^2} = \frac{1-(c_z+r_z)^2}{t_x^2} = \frac{1-(c_z+r_z)^2}{1-t_y^2} = \frac{(c_x-r_x)^2}{t_x^2}\end{aligned}$$

In summary, we have shown

$$1 \leq \frac{\theta_x(x,y)}{-x} \leq \frac{c_x - r_x}{t_x}.$$

3.3 Deriving the Bound

Now we combine the bounds for s_y and $\frac{\theta_x(x,y)}{-x}$:

$$1 \leq \frac{\theta_x(x,y)}{-x s_y(\sqrt{1-x^2-y^2})} \leq \frac{\theta_x(x,y)}{-x}\sqrt{\frac{r_z-c_z}{r_z}} \leq \frac{c_x-r_x}{t_x}\sqrt{\frac{r_z-c_z}{r_z}}$$

Finally, we want to express this bound solely in terms of x-coordinates. The points $(c_x - r_x, 0, c_z + r_z)^\mathsf{T}$, $(t_x, 0, 0)^\mathsf{T}$ and $(c_x + r_x, 0, c_z - r_z)^\mathsf{T}$ all lie in the plane of the circle $\partial\mathbb{D}$. Therefore, they also lie on a common line in the xz-plane, i.e.

$$\frac{-2r_x}{2r_z} = \frac{t_x - (c_x + r_x)}{0 - (c_z - r_z)} \qquad \Rightarrow \qquad \frac{r_z - c_z}{r_z} = \frac{t_x - (c_x + r_x)}{-r_x}.$$

Rewriting the upper bound in this manner, we conclude

$$\frac{\max_{\xi_0,\xi_1\in[0,1]} p(\xi_0,\xi_1)}{\min_{\xi_0,\xi_1\in[0,1]} p(\xi_0,\xi_1)} \leq \frac{c_x - r_x}{t_x}\sqrt{\frac{t_x - (c_x + r_x)}{-r_x}}. \tag{4}$$

3.4 Analyzing the Bound

It is easy to evaluate the bound in Equation (4) for any given spherical cap but we would prefer general guarantees on the quality of our samples. Thus, we seek a more conservative upper bound that only depends on the radius r_y of the spherical cap $\mathbb{D}$. This radius is directly related to the distance to the center of the light source through $r_y = \frac{r}{\|\boldsymbol{d}\|_2}$. Hence, it characterizes a sphere around the light source.

We could continue to derive bounds analytically but there is a much simpler way. Through the computations above, we have obtained bounds that are independent of the random numbers ξ_0, ξ_1. The set of spherical caps modulo rotation around the surface normal $\boldsymbol{n}$ is only two-dimensional. We can effortlessly consider the set of all spherical caps by sampling it densely and computing the bound.

The formulation of the bound even suggests a specific way to do this. It only depends on $c_x - r_x$, $c_x + r_x$ and t_x and one of the three quantities is redundant. As explained in Section 3.3, $(c_x - r_x, c_z + r_z)^\mathsf{T}$, $(t_x, 0)^\mathsf{T}$ and $(c_x + r_x, c_z - r_z)^\mathsf{T}$ lie on a common line. Besides two of these points lie on the unit circle, i.e.

$$c_z + r_z = \sqrt{1 - (c_x - r_x)^2}, \qquad c_z - r_z = -\sqrt{1 - (c_x + r_x)^2}.$$

By computing the intersection of the line through $(c_x - r_x, c_z + r_z)^\mathsf{T}$ and $(c_x + r_x, c_z - r_z)^\mathsf{T}$ with the x-axis, we obtain t_x.

Furthermore, we know

$$-1 < c_x + r_x < t_x < c_x - r_x < 1 \quad \text{and} \quad 0 < t_x. \tag{5}$$

Therefore, we sample $c_x + r_x$ uniformly in $[-1, 1]$ and $c_x - r_x$ uniformly in $[0, 1]$ using 32,768 samples for each. Then we construct a two-dimensional grid, compute t_x and discard samples where one of the Inequalities (5) is violated. For each sample, we also compute

$$r_y = \sqrt{r_x^2 + r_z^2}.$$

For any given threshold on r_y, we consider all samples where r_y is below the threshold and determine the maximal value of the bound in Equation (4). By doing this for a dense sampling of thresholds, we obtain the sought-after conservative bound, which is shown in Figure 2.

We note that there is a crease in the graph at $\frac{1}{r_y} = 2$. From this point onward, the upper bound is constant at $\sqrt{2}$. This is the case because spherical caps, no matter how small, always realize a quotient of $\sqrt{2}$ in their densities as they vanish below the horizon:

$$\lim_{c_x \to 1 + r_x} \frac{c_x - r_x}{t_x}\sqrt{\frac{t_x - (c_x + r_x)}{-r_x}} = \frac{1}{1}\sqrt{\frac{1 - (1 + 2r_x)}{-r_x}} = \sqrt{2}$$

The bound goes to infinity as $\frac{1}{r_y} \to 1$, i.e. for shading points on the surface of the light source. However, it drops off quickly from there. For example, for $\frac{1}{r_y} \geq 1.094$ the bound is below two.

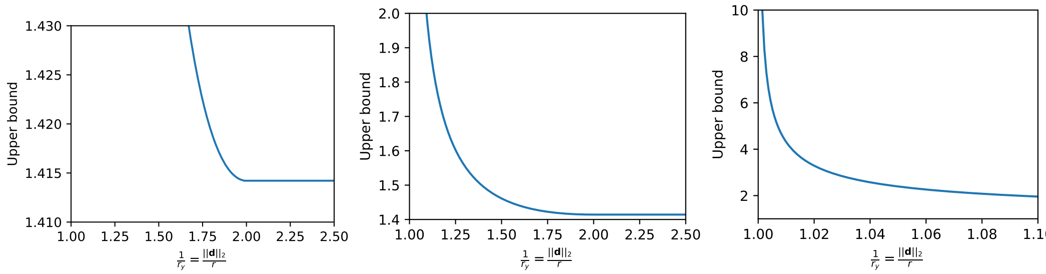

Fig. 2. A variant of the upper bound in Equation (4) that only depends on r_y. All three plots show the same graph with different limits for the x- and y-axis. The x-axis corresponds to $\frac{1}{r_y}$ which is an intuitive quantity as it is proportional to the distance to the light source center. For $\frac{1}{r_y} = 1$, the shading point is on the surface of the spherical light source, for $\frac{1}{r_y} = 2$ its distance to the light source surface matches the radius of the spherical light source.