Fast Approximation of Multiple Scattering in Inhomogeneous Participating Media

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Abstract

This paper presents a fast approximation for the solution of the radiative transfer equation in inhomogeneous participating media. This scheme traces rays from the point light creating spherical wavefronts. While we march on a particular ray, we assume that other rays find similar material properties, which allows the solution of the diffusion equation for this ray. The proposed approximation of the scattered component can be computed in parallel with obtaining the direct term at practically no additional cost. The approximation may be used for direct visualization, an initial guess for an iterative solution, or for the main part of a Monte Carlo solution.

1. Introduction

Rendering participating media is one of the most challenging problems of realistic illumination. Beyond computer graphics, the problem is important in many other fields, such as in engineering simulations, nuclear technology, radiotherapy, PET/SPECT reconstruction, etc. The change of radiance $L(\vec{x},\vec{\omega})$ at point $\vec{x}$ and in direction $\vec{\omega}$ in participating media is expressed by the *radiative transport equation*:

$$\vec{\omega}\cdot\vec{\nabla}L = \left.\frac{dL(\vec{x}+\vec{\omega}s,\vec{\omega})}{ds}\right|_{s=0} =$$

$$-\sigma_t L(\vec{x},\vec{\omega}) + \sigma_s \int_{\Omega'} L(\vec{x},\vec{\omega}')P(\vec{\omega}',\vec{\omega})\, d\omega', \qquad (1)$$

where σ_t is the *extinction coefficient* describing the probability of collision in a unit distance. When collision happens, the photon is either scattered or absorbed, so the extinction coefficient is broken down to *scattering coefficient* σ_s and *absorption coefficient* σ_a:

$$\sigma_t = \sigma_a + \sigma_s, \quad a = \frac{\sigma_s}{\sigma_t}$$

where a is the probability of reflection given that collision occurred, and is called the *albedo*. The probability density of the reflected direction is defined by *phase function* $P(\vec{\omega}',\vec{\omega})$. In homogeneous media volume properties σ_t, σ_s, and $P(\vec{\omega}',\vec{\omega})$ do not depend on position $\vec{x}$. In inhomogeneous media these properties depend on the actual position.

Cerezo et al. [CPP+05] classified algorithms solving the transfer equation as analytic, stochastic, and iterative.

Analytic techniques rely on simplifying assumptions, such as that the volume is homogeneous, and usually consider only the single scattering case [Bli82, SRNN05]. Jensen et al. [JMLH01] attacked the subsurface light transport by assuming that the space is partitioned into two half spaces with homogeneous material and developed the dipole model. Narasimhan and Nayar [NN03] proposed a multiple scattering model for optically thick homogeneous media and isotropic light source.

Stochastic methods apply Monte Carlo integration to solve the transport problem [KH84, JC98]. These are the most accurate but are too slow in interactive applications.

Iterative techniques need to represent the current radiance estimate that is refined in each step. The radiance function is specified either by finite-elements, using, for example, the zonal method [RT87], spherical harmonics [KH84], radial basis functions [ZRL+08], metaballs, etc.

Stam [Sta95] introduced *diffusion theory* to compute energy transport. Here the angular dependence of the radiance is approximated by a two-term expansion:

$$L(\vec{x},\vec{\omega}) \approx \tilde{L}(\vec{x},\vec{\omega}) = \frac{1}{4\pi}\phi(\vec{x}) + \frac{3}{4\pi}\vec{E}(\vec{x})\cdot\vec{\omega}.$$

By enforcing the equality of the averages and the direction weighted averages of the original and the approximated functions, respectively, we get the following equations for

fluence ϕ and *vector irradiance* $\vec{E}(\vec{x})$:

$$\phi(\vec{x}) = \int_{\Omega} L(\vec{x},\vec{\omega})\, d\omega, \quad \vec{E}(\vec{x}) = \int_{\Omega} L(\vec{x},\vec{\omega})\vec{\omega}\, d\omega.$$

Substituting this two-term expansion into the radiative transfer equation and averaging it for all directions, we obtain the following diffusion equations:

$$\vec{\nabla}\phi(\vec{x}) = -3\sigma_t'\vec{E}(\vec{x}), \quad \vec{\nabla}\cdot\vec{E}(\vec{x}) = -\sigma_a\phi(\vec{x}). \qquad (2)$$

where σ_t' is the *reduced extinction coefficient*:

$$\sigma_t' = \sigma_t - \sigma_s g, \quad g = \int_{\Omega'} (\vec{\omega}'\cdot\vec{\omega})P(\vec{\omega}'\cdot\vec{\omega})\, d\omega'.$$

Parameter g describes the anisotropy of the material.

In [Sta95] the diffusion equation is solved by iteration using either a multi-grid scheme or a finite-element blob method. Geist et al. [GRWS04] computed multiple scattering as a diffusion process, using a lattice-Boltzmann solution method.

This paper presents a fast approximation for the solution of the radiative transfer equation. We shall assume that the primary source of illumination is a single point light source in the origin of our coordinate system. More complex light sources can be modeled by translation and superposition. The fast approximation is governed by the diffusion theory, where the single pass approximate solution is made possible by assumptions that the medium is locally homogeneous and spherically symmetric. The solution is approximate, similarly to the application of a coarse grid, but can be obtained in parallel with the single scattering term, practically at no additional cost.

1.1. The importance of good, approximate solutions

Approximate solutions can be used not only for fast visualizations, but also to speed up more precise solutions of the transport equation, letting the precise method focus on only the "rest" of the problem.

In case of an iteration solution, the error at a particular step n can be upperbound in the following way:

$$\|L_n - L\| \le \lambda^n \|L_0 - L\|,$$

where L is the real solution, L_n is the current guess, λ is the contraction of the transport operator, and L_0 is the initial guess. Note that the error is proportional to the norm of the difference of the initial guess L_0 and the final solution L. Thus, having a good initial guess that is not far from the solution, the error can be significantly reduced.

On the other hand, if we use a Monte Carlo solution, then *main part separation* can be applied, which expresses the radiance as the sum of initial guess L_0 and some residual ΔL. Substituting this decomposition into the original equation, we obtain an equation that is similar to the original one, but its source term and boundary conditions will be equal to the error of the initial guess. Thus, if the initial guess is good, the integrands will have much smaller variation.

2. The method of homogeneous spherical layers

Let us consider just a single beam starting at the origin where the point source of radiant intensity Φ is. When a beam is processed, we shall assume that other beams face to the same material characteristics, i.e. we assume that the scene is *spherically symmetric*. Note that the assumption on spherical symmetry does not mean that only one beam is processed. We take many beams originating from the source, and each of them are traced independently assuming that other rays face the same material properties as the current beam.

In case of spherical symmetry, the radiance of the inspected beam at point $\vec{x}$ and in direction $\vec{\omega}$ may depend just on distance $r = |\vec{x}|$ from the origin and on angle θ between direction $\vec{\omega}$ and the direction of point $\vec{x}$. The unit direction vector of point $\vec{x}$ will be denoted by $\vec{\omega}_{\vec{x}} = \vec{x}/|\vec{x}|$. This allows parametrization $L(r,\theta)$ instead of $L(\vec{x},\vec{\omega})$. The fluence depends just on distance r and vector irradiance $\vec{E}(\vec{x})$ has the direction of the given point, that is $\vec{E}(\vec{x}) = E(r)\vec{\omega}_{\vec{x}}$.

Expressing the divergence operator in spherical coordinates, we get:

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = \vec{\nabla}\cdot(E(r)\vec{\omega}_{\vec{x}}) = \frac{1}{r^2}\frac{\partial(r^2E(r))}{\partial r}.$$

Thus, the scalar versions of the diffusion equations are:

$$\frac{d\phi(r)}{dr} = -3\sigma_t'E(r), \quad \frac{1}{r^2}\frac{d(r^2E(r))}{dr} = -\sigma_a\phi(r). \qquad (3)$$

If we have a point light source, then this equation has a singularity at $r = 0$ where the radiance gets infinite. To solve this problem, we rewrite the equations to use power ψ instead of the fluence. In case of spherical symmetry, at distance r the power is computed on area $4r^2\pi$, thus the correspondences between the fluence and the vector irradiance with the power measures are $\psi_0 = r^2\phi$ and $\psi_1 = r^2E$ (note that the correspondence between the radiance and the fluence already includes the 4π factor). Substituting these into the differential equation we obtain:

$$\frac{d\psi_0(r)}{dr} = \frac{2}{r}\psi_0 - 3\sigma_t'\psi_1(r), \quad \frac{d\psi_1(r)}{dr} = -\sigma_a\psi_0(r). \qquad (4)$$

For homogeneous infinite material, the differential equation can be solved analytically:

$$\psi_0^h(r) = Ae^{-\sigma_e r}r,$$

$$\psi_1^h(r) = \frac{2}{3r\sigma_t'}\psi_0(r) - \frac{1}{3\sigma_t'}\frac{d\psi_0(r)}{dr} = \frac{A}{3\sigma_t'}e^{-\sigma_e r}(\sigma_e r + 1). \qquad (5)$$

where $\sigma_e = \sqrt{3\sigma_a\sigma_t'}$ is the *effective transport coefficient*, and A is an arbitrary constant that should be determined

from the boundary conditions. According to the first equation $\psi_0(0) = 0$, thus only the second equation is free at the boundary. The radiant intensity of the source is Φ, which is made equal to the vector irradiance:

$$\psi_1(0) = \frac{A}{3\sigma_t'} = \Phi \quad \Longrightarrow \quad A = 3\sigma_t'\Phi.$$

2.1. Initial approximation with wavefront tracing

With equation 4 we established two differential equations that describe the power evolving as we move along a ray started at the origin. These equations can be solved by numerical integration while marching on the ray and taking samples from the material properties $\sigma_t(r)$ and $\sigma_s(r)$.

In order to obtain the initial values, we take the solution for homogeneous material:

$$\psi_0(0) = \psi_0^h(0) = 0, \quad \psi_1(0) = \psi_1^h(0) = 3\sigma_t'\Phi.$$

Care should be practiced when starting the iteration. At the beginning $\psi_0 = 0$ and $r = 0$, so when evaluating $\frac{2}{r}\psi_0$, we have a $0/0$ type undefined value. Using again the solution of the homogeneous case

$$\lim_{r\to 0} \frac{2}{r}\psi_0 = \lim_{r\to 0} \frac{2}{r} 3\sigma_t'\Phi e^{-\sigma_e r} r = 6\sigma_t'\Phi.$$

As ray marching proceeds taking steps Δ increasing distance r, material properties σ_t, σ_s, and g are fetched at the sample location, and state variables $\psi_0[n]$, and $\psi_1[n]$ are updated according to the numerical quadrature, resulting in the following iteration formula for step n:

$$\begin{aligned} \psi_0[n] &= \psi_0[n-1]\left(1 + \frac{2\Delta}{r}\right) - 3\sigma_t'\psi_1[n-1]\Delta, \\ \psi_1[n] &= \psi_1[n-1] - \sigma_a\psi_0[n-1]\Delta. \end{aligned} \tag{6}$$

In order to execute wavefront tracing, the volume is resampled to a new grid that is parameterized with spherical coordinates. A voxel of the new grid with (u,v,w) coordinates represents point

$$R(w\cos\alpha\sin\theta, w\sin\alpha\sin\theta, w\cos\theta),$$

where $\alpha = 2\pi u$, $\theta = \arccos(1-2v)$, and R is the size of the volume. Note that this parametrization provides uniform sampling in the directional domain. A (u,v) pair encodes the ray direction, while w encodes the distance from the origin. This texture is processed w-layer by w-layer, i.e. stepping the radius r simultaneously for all rays. In a single step the GPU updates powers according to equation 6.

At the end of wavefront tracing, we return to the original grid and compute the approximate radiance values of the grid points. The radiance of sample point $\vec{x}$ being in direction $\vec{\omega}_{\vec{x}}$ and at distance r with respect to the origin, assuming view direction $\vec{\omega}$ is

$$L(\vec{x},\vec{\omega}) \approx \frac{\psi_0(r)}{4\pi r^2} + \frac{3\psi_1(r)}{4\pi r^2}(\vec{\omega}_{\vec{x}}\cdot\vec{\omega}).$$

3. Results

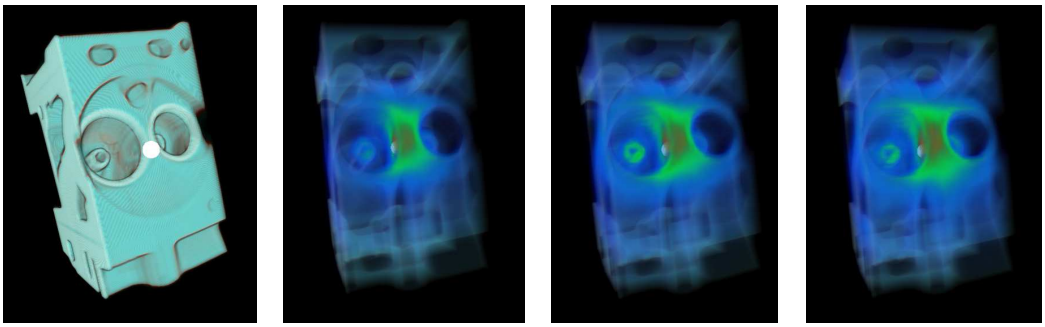
The proposed method has been implemented in CUDA and used as an initial phase of a distributed iterational solver. In the resulting application the user can place the point source interactively in the volume and the system reacts to the user changes in real-time. The initial distribution of both the direct term and the approximated total radiance in an $128\times128\times64$ volume required 30msec on an NVIDIA GeForce 8800 GTX GPU. Figures 3 and 4 show the density of the volume and the superimposed color coded radiance approximation in two different datasets. We compared the initial approximation to the direct term and to the global illumination solution. Note that our initial approximation is quite close to the final global illumination solution and much better than computing only the direct term. However, its computational cost is similar to that of the direct term generation. In order to get the same accuracy, the required number of iterations of the global illumination solver starting from the proposed initial estimation is just 50% of the number of iterations without the estimation and initiating the iteration from the direct term. In other words, the speed of the global illumination solution was doubled.

4. Conclusions

This paper proposed a simple but effective method to approximately distribute the scattered radiance caused by point light sources. Actually, a modified ray marching algorithm is executed that transports the two-term expansion of the diffusion term. Comparing to classical methods that would distribute only the direct lighting term in the first pass, the new method provides much better results at the same computational cost. The problem of distributing only the direct term is that its intensity caused by a point light source decreases with the square of the distance, while — as shown by the diffusion solution — the radiance is expected to decrease with only the distance, thus the direct term is practically zero far from the light source.

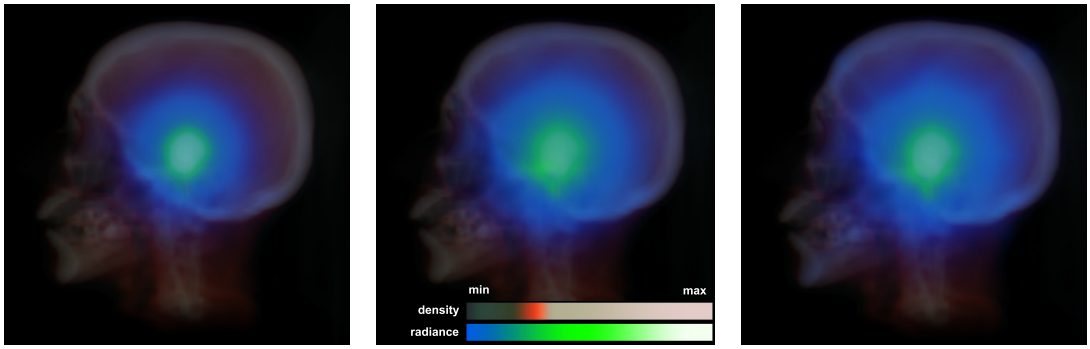
Comparing to initializing the radiance with the homogeneous diffusion solution, we should emphasize that the proposed wavefront tracing can handle strongly heterogeneous materials as well. Suppose, for example, that the volume has "holes", i.e. regions enclosed by strongly absorbing material. The homogeneous diffuse solution would ignore the local material properties, thus this region would also be initialized with significant radiance. However, in our approach, the radiance approximation takes into account the material properties having seen by rays before arriving at a particular point, thus this regions would have small radiance.

The quick initial solution can be visualized directly, or can be used as a good guess for iterative refinement or a main part for main part separation type Monte Carlo methods. In the shown models, the proposed initial estimation doubled the speed of the global illumination solver.



Density only | Direct term | Proposed approximation | Global illumination solution

Figure 1: *Efficiency of the initial approximation for the engine dataset. Note that the proposed approximation is much better than the direct term. The radiance is color coded to emphasize the differences.*



Direct term | Proposed approximation | Global illumination solution

Figure 2: *Efficiency of the initial approximation for the head dataset. Note that the proposed approximation is quite close to the final global illumination solution. The radiance is color coded to emphasize the differences.*

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